Hendecagonal Near-miss Polyhedral Cages

Jonathan G. Heddle[1], Agnieszka Kowalczyk[2] and Bernard M. A. G. Piette[3]

[1]Bionanoscience and Biochemistry Laboratory, Malopolska Centre of Biotechnology, Jagiellonian University, Kraków, Poland; jonathan.heddle@uj.edu.pl
[2]Faculty of Mathematics and Computer Science & Bionanoscience and Biochemistry Laboratory, Malopolska Centre of Biotechnology, Jagiellonian University, Kraków, Poland; agnes.kowalczyk@student.uj.edu.pl
[3]Dept. of Mathematical Sciences, Durham University, UK; b.m.a.g.piette@durham.ac.uk

Abstract

Inspired by the concept of near-misses (namely near-miss Johnson a.k.a. NMJ solids) we investigate a related family of shapes which we call polyhedral near-miss cages. In this paper we discuss a few possible spherical arrangements of (nearly-)regular hendecagons. We believe that it may be of potential use in biotechnology and design as well as an inspiration for future artwork.

Introduction

There is no precise definition of the so-called near-miss Johnson solids hence no general theory is established. They are commonly described as polyhedra whose faces are so close to regular that they can often be physically constructed without noticing the distortion [4]. There was, however, an attempt to list (according to certain rules) all near-miss Johnson solids known so far [6]. Here we lessen the NMJ constraints and impose that only some of the faces have to be regular or nearly-regular. In this way we develop the idea of polyhedral cages, the connected sphere-like 2-surfaces made out of N planar congruent regular polygons glued edge-to-edge with the restriction that the glued edge must be preceded and followed by unglued edges of the same face (i.e. no two consecutive face edges are glued to another face or faces). Moreover, two neighbour faces can only be glued together once and each polygon needs to have at least three adjacent faces. As our focus is set on the cage faces, we neglect the regularity of the occuring holes. However, when aiming at the precise combinatorial structure, gluing faces together might distort them geometrically so that the regularity is no longer preserved. This is what we shall call a near-miss polyhedral cage – a polyhedral cage whose faces are nearly regular polygons. We can therefore calculate the distortion level and, if the error is tiny (say less than 2%), include the structure in the library of near-miss cages. A polyhedral cage with zero error we call an exact cage (its faces are regular polygons).

Polyhedral Cages

Our starting point is the structure presented in Figure 1 made out of 24 hendecagons [5]. When the holes are filled in (with triangles and quadrangles) one obtains the polyhedral surface of a near-miss Johnson solid (cf. [3], and especially in [8]) – it is a well-known result that a convex polyhedron (other than prism and antiprism) whose faces include hendecagons cannot be a regular-faced polyhedron. This follows from the theorem by Johnson and Grünbaum [1]. The construction of this polyhedron is based on the snub cube whose vertices correspond to the optimal distribution of 24 points on a sphere (optimal here means that the minimal distance between the points is as large as possible) [7]. This near-miss polyhedral arrangement of 24 hendecagons brings us to explore other possible hendecagonal cages with tiny distortions from regularity.

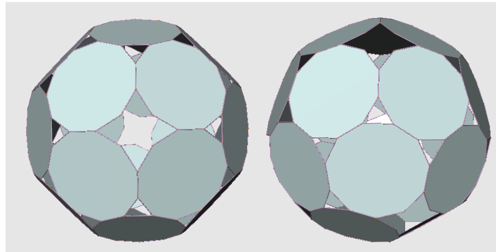
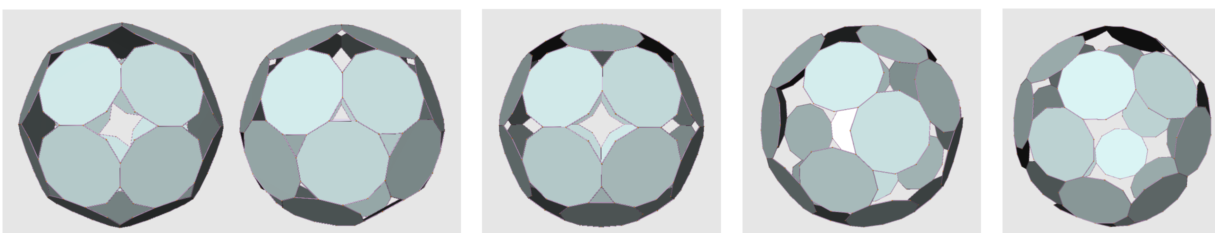


Figure 1: $[1,1,1,1,2]_{24}$ – *a cage of 24 hendecagons based on the symmetry of the snub cube with* $6 \times (2-2-2-2)$ *and* $32 \times (1-1-1)$ *holes; both top and side view are presented.*

In this section we show four polyhedral arrangements of 24 hendecagons (Fig.2) different from the one presented in Figure 1 and three arrangements of 12 hendecagons (Fig.3). For convenience, in this paper, we shall label the cages with a sequence of integers describing the unglued edges distribution among faces (anticlockwise). For instance, the configuration from Figure 1 is called $[1,1,1,1,2]_{24}$ which means that each of the 24 faces is adjacent to 5 holes and contributes 1, 1, 1, 1, 2 edges to those holes respectively, whereas the $[1,2,2,2]_8 + [1,1,1,1,2]_{16}$ cage from Figure 2a has 8 faces adjacent to four holes each contributing to 1, 2, 2, 2 edges and 16 faces adjacent to five holes each contributing to 1, 1, 1, 1, 2 edges.

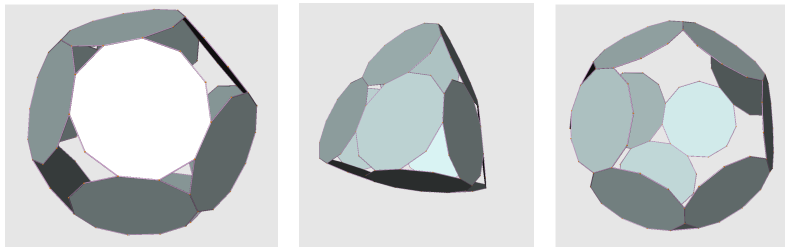


(a) $[1,2,2,2]_8 + [1,1,1,1,2]_{16}$ **(b)** $[1,2,2,2]_{24}$ **(c)** $[2,2,4]_{24}$ **(d)** $[2,3,3]_{24}$

Figure 2: *Examples of polyhedral cages with 24 hendecagons and:* *(a)* $10 \times (2-2-2-2)$, $24 \times (1-1-1)$ *holes (both top and side view are presented),* *(b)* $18 \times (2-2-2-2)$, $8 \times (1-1-1)$ *holes, (c)* $8 \times (2-4-2-4-2-4)$, $6 \times (2-2-2-2)$ *holes, (d)* $8 \times (3-3-3-3-3-3)$, $6 \times (2-2-2-2)$ *holes.*

The main features that distinguish the configurations from each other (apart from the number of polygons) are the hole shapes and the number of neighbouring faces. We encode the hole shape using the following notation (e_1-e_2-...-e_n), where each consecutive integer e_i is the number of edges contributed from the i-th face adjacent to that hole (in anticlockwise direction). Notice also that some of the cages are chiral, meaning that they are not congruent with their mirror image. Interestingly, all of those cages but one, in Figure 2a, have equivalent faces, i.e. every face can be mapped onto any other face by means of proper rotations. This above all means that each face has the same number of neighbouring faces, same neighbouring holes of (possibly) different types and the same distribution of unglued edges among the holes.

The table below lists some properties of the presented cages. Only one of them – $[2,3,3]_{24}$ – is an exact cage, whereas the rest are near-miss cages. The diameters were calculated assuming unit target edge length as defined in the next section. The underlying topological structure of a cage is the polyhedron created by joining the centres of the cage faces. These topological structures match those of Archimedean solids for all the cages except the $[1,2,2,2]_8 + [1,1,1,1,2]_{16}$ cage for which we have the gyroelongated square bicupola (Johnson solid J45).



(a) $[1, 2, 1, 3]_{12}$ **(b)** $[1, 2, 2, 2]_{12}$ **(c)** $[1, 3, 4]_{12}$

Figure 3: *Examples of near-miss polyhedral cages with 12 hendecagons and (a)* $6 \times (2-3-2-3)$, $8 \times (1-1-1)$ *holes, (b)* $6 \times (2-2-2-2)$, $4 \times (2-2-2)$, $4 \times (1-1-1)$ *holes, (c)* $4 \times (3-4-3-4-3-4)$, $4 \times (1-1-1)$ *holes.*

Table 1: *Hendecagonal cages characterization (rd = relative error).*

	$[1, 1, 1, 1, 2]_{24}$	$[1, 2, 2, 2]_8$+ $[1, 1, 1, 1, 2]_{16}$	$[1, 2, 2, 2]_{24}$	$[2, 2, 4]_{24}$	$[2, 3, 3]_{24}$	$[1, 2, 1, 3]_{12}$	$[1, 2, 2, 2]_{12}$	$[1, 3, 4]_{12}$
cage diameter	9.19	9.37	9.64	10.39	11.02	6.90	7.06	8.03
angle rd	0.27%	1.15%	0.11%	0.40%	0.00%	1.80%	1.49%	0.50%
length rd	0.50%	1.34%	0.00%	0.32%	0.00%	1.58%	0.01%	0.53%
numbers of neighbours	5	4 or 5	4	3	3	4	4	3
underlying topological structure	snub cube	J45	rhombi-cubocta-hedron	truncated octa-hedron	truncated octa-hedron	cubocta-hedron	cubocta-hedron	truncated tetra-hedron
equivalent faces	yes	no	yes	yes	yes	yes	yes	yes
chiral	yes	yes	no	yes	no	no	no	yes

Optimization Process – Short Description

The geometry of the assemblies presented in the previous section was refined by a custom-written C++ program that used Monte-Carlo minimization of an energy functional of the following form:

$$E = c_e \sum_{faces} \sum_{edges} (l_i - L)^2 + c_a \sum_{faces} \sum_{corners} (a_i - A)^2 + c_p \sum_{faces} P(\text{face}).$$

The first term measured by how much the edge lengths l_i differ from a chosen reference length L. The second one measured by how much the angle between the face edges, a_i, deviates from the angle of the regular polygon, A. The last term, where P(face) is a somewhat complex expression involving all the vertices of each face, enforced the planarity of the faces and the parameter c_p was chosen to be several orders of magnitude larger than c_e and c_a. We also imposed local convexity of the cage to avoid concave structures.

To construct a cage, we started by describing how the faces are linked together and which of their edges are part of holes. As a starting point, the vertices were very approximately distributed on a sphere using the face initial configuration as a guide. We then used the Metropolis algorithm, moving randomly each vertex, to minimise the functional above. A detailed description of the algorithm used will be reported elsewhere [2].

It is worth mentioning that the number of potential configurations is very large, but very few lead to nearly regular cages. We obtained many other hendecagonal cages, but they either had much bigger distortions or poor overall symmetry. Most importantly, the refined cages are not unique – it is possible to trade off the length and angle deformations against each other by adjusting the parameters c_e and c_a in the above equation. Though it is not always feasible, we try to list the structures with comparable deviations for both edges and angles. Sometimes we are able to reduce one of the deviations to almost zero but not the other which suggests that for some combinatorial structures one cannot mitigate the deviation level.

Summary and Conclusions

The results presented in this paper are just a small part of a long-run project aiming at the classification of the polyhedral cages. The concept of near-miss polyhedral cages is an attempt to generalize the idea of NMJs and might be a starting point for drawing more universal conclusions about NMJs.

Acknowledgements

We want to thank Symfonia team members from Bionanoscience and Biochemistry Laboratory (BBL) as well as Craig S. Kaplan for a fruitful collaboration and discussions. JGH and AK were funded by National Science Centre (NCN, Poland) grant no. 2016/20/W/NZ1/00095 (Symfonia-4) awarded to Jonathan Gardiner Heddle. BMAGP would like to thank the BBL for its hospitality. The figures were produced with the Antiprism Polyhedron Modelling Software (https://www.antiprism.com/).

References

[1] B. Grünbaum and N. W. Johnson. "The faces of a regular-faced polyhedron." *J. London Math. Soc.*, 40 (1965), pp. 577–586.

[2] J.G. Heddle, A. Kowalczyk and B.M.A.G. Piette. In preparation.

[3] C. S. Kaplan. Near Misses. [online, accessed 1 March 2019] http://www.cgl.uwaterloo.ca/csk/projects/nearmisses/.

[4] C. S. Kaplan and G. W. Hart. "Symmetrohedra: Polyhedra from Symmetric Placement of Regular Polygons." *Bridges Conference Proceedings*, Winfield, Kansas, USA, Jul. 27–29, 2001, pp. 21–28. https://archive.bridgesmathart.org/2001/bridges2001-21.pdf.

[5] A. D. Malay et al. "An ultra-stable gold-coordinated protein cage displaying reversible assembly." *Nature*, 569 (2019), pp. 438-442.

[6] J. McNeill. Miscellaneous Polyhedra: Johnson Solid Near Misses. [online, accessed 1 March 2019] http://www.orchidpalms.com/polyhedra/acrohedra/nearmiss/jsmn.htm.

[7] R. M. Robinson. "Arrangement of 24 points on a sphere." *Math. Annalen*, 144 (1961), pp. 17–48.

[8] Wikipedia page. [online, accessed 1 March 2019] https://bit.ly/2VnvBwM.