Divisible Skylines: Exploring Least Common Multiples and Divisibility through Visual Art

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Abstract

We present an alternative way to consider number theoretic concepts through visual art. Our visualization method, Divisible Skylines, is an artistically motivated study of least common multiples. It demonstrates how beauty and mathematical understanding can join hands in the study of divisibility. We present original artwork based on our method, examine mathematical properties of Divisible Skylines through the artwork, and point out several artistically interesting visual aspects. Our method opens possibilities for developing playful and creative ways to teach divisibility and number theory. Divisible Skylines offer interest for artists, educators and students alike.

Introduction

The basis of this paper is the artwork *Divisible Dreams* by Saara Lehto (Figure 1). The creative process behind this work entwines mathematics and visual art in a natural and intriguing way. Divisibility has previously been embodied with rhythms [3], but not many visual representations can be found. We present a visualization method that is both mathematically illustrative and artistically pleasing.

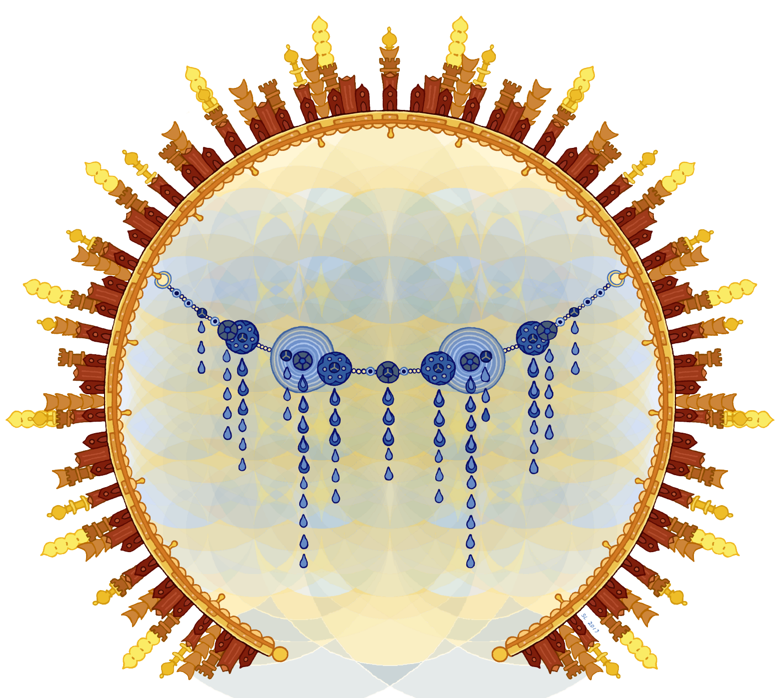

Figure 1: *Divisible Dreams by Saara Lehto, 2019*

In the heart of *Divisible Dreams* is a visualization method designed for finding least common multiples. In this paper we will explain this method and discover how number theoretic concepts and artistically interesting questions intertwine in the exploration of the artwork.

The idea for *Divisible Dreams* originated at a Maths in Motion [1] training event in Ommen, Netherlands in February 2018. Participants tried out a classic activity by Dr. Schaffer and Mr. Stern where two people clap their names (of different lengths) and by doing so produce different rhythms and an understanding for the concept of least common multiples (lcm) [3]. Participants were then asked to demonstrate what they had learned using building blocks. One group created an interesting pattern (Figure 2). We were fascinated by this idea and wanted to further explore the artistic possibilities of the representation.

Figure 2: *Finding* lcm *of 2, 5 and 6 using building blocks*

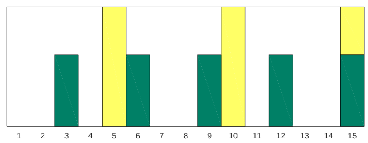

Figure 3: *Divisible Skyline for 3 and 5 generated by divskyline [4]*

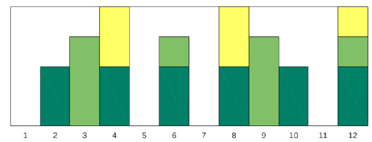

Figure 4: *Divisible Skyline for 2, 3 and 4 generated by divskyline [4]*

Divisible Skylines: Visualizing Least Common Multiples

The *least common multiple* of positive integers n and m is the smallest positive integer divisible by both n and m. It is denoted by $\mathrm{lcm}(n,m)$. The *greatest common divisor* $\gcd(n,m)$ of n and m is the largest integer that divides both n and m. These concepts are connected by the formula $\mathrm{lcm}(n,m) = nm/\gcd(n,m)$. Both definitions can be extended for integers $n_1,\ldots,n_k$ in the natural way.

To find the least common multiple of, say, 3 and 5, we can draw markers for 3, 6, 9 and so on and for 5, 10, 15 and so on and look where the markers first meet. In our method we draw pillars of different heights to mark the multiples of 3 and 5. From Figure 3 we can see that $\mathrm{lcm}(3,5) = 15$.

We found that visualizing least common multiples this way creates different pleasing patterns that can be viewed as landscapes, bridges, scattered trees on a hill, icicles forming on eaves of buildings or—as we often found was the case for us—city skylines full of towering buildings of different heights. We thus call our visualization method *Divisible Skylines*.

Different combinations of integers produce different skylines. We are interested in how the number theoretic properties of given integers relate to the visual properties of their skyline, and whether our visualization method could be modified to produce artistically or mathematically more interesting results.

In general, *Divisible Skylines* are created by drawing pillars—the towers of our skyline—to represent the multiples of given integers: For any integer n we define an n-tower to be a pillar of width 1 and height n. Consider places $1,\ldots,\mathrm{lcm}(n,m)-1$ on the x-axis, where n and m are positive integers. For the skyline of n and m we build n-towers on all the places that are multiples of n and we build m-towers on all the places that are multiples of m. For artistic reasons we have defined *Divisible Skylines* not to include the last place $\mathrm{lcm}(n,m)$. We could just as well have included it, which is, in fact, what we have done in Figures 3 and 4. Figure 4 demonstrates the natural way *Divisible Skylines* can be defined for integers $n_1,\ldots,n_k$.

Figures 3 and 4 are produced by an Octave/Matlab function called divskyline by Tommi Sottinen [4]. Figure 3 is produced by entering divskyline([3, 5]) at the prompt and Figure 4 is produced by entering divskyline([2, 3, 4]). Divskyline was used as a tool in the creation of *Divisible Dreams*, and we are interested in developing it further to include a more user friendly platform and to enable more artistic possibilities.

The Beauty of Number Theory Made Visible

If you play with divskyline [4] online, draw your own skylines or just look at any skyline pictures, you become quickly aware of several visually interesting patterns. As you look at the towers more carefully, you can see how these phenomena are in fact driven by the integers generating the picture. The beauty you can see is in fact mathematics, specifically number theory [2].

In the following, we will connect some number theoretical properties with the visual properties of the skylines. We will use as an example the outside circle of *Divisible Dreams* (Figure 1) which depicts a curved skyline of integers $2, 3, 4, 5, 6$ and 8 with the help of differently styled towers.

Drawing skylines one becomes very quickly convinced that they are all **bilaterally symmetric**. Why is that? One of the first properties one learns about divisibility is that h is divisible by n if and only if $nm - h$ is divisible by n for any integer m. If we place here n as any of the generating integers of a skyline and m as the least common multiple of a skyline, we see that this property guarantees that all skylines are symmetric: an n-tower appears on place h if and only if it also appears also on the symmetric place $nm - h$. The symmetry can be also visually confirmed by considering any skyline: all integers $n_1, \ldots, n_k$ would have towers in places 0 and $\mathrm{lcm}(n_1, \ldots, n_k)$ just outside each end of the skyline and thus towers drawn starting from either endpoint into either direction yield the exact same pattern. For example in Figure 1 starting from either end 2-towers appear every second place, 3-towers every third place and so on.

When drawing a skyline for more than two integers, sometimes two or more towers appear **in the same place**. In Figure 1, you notice that the 6-tower never appears alone, it is always joined by a 2-tower and a 3-tower. This is because 6 is divisible by both 2 and 3. In general an n_i-tower and an n_j-tower are in the same place if and only if that integer is divisible by both of them, namely, by $\mathrm{lcm}(n_i, n_j)$. Furthermore, if n_i divides n_j, then every n_j-tower always has an n_i-tower on the same place.

In Figure 1, we see that the 6- and 8-towers are sometimes in the same place, but never in **adjacent places**. Visually we can understand that they will never be in adjacent places, as both are always together with a 2-tower and two 2-towers can never be side by side. However, 5-towers and 6-towers are frequently in adjacent places and so are 3-towers and 5-towers. The mathematical explanation is that the Diophantine equation $ax + by = c$ where $a, b, c \in \mathbb{Z}$ has integer solutions x and y if and only if c is divisible by $\gcd(a, b)$. This gives us an easy way to check whether two integers on a skyline have common factors: Two towers appear side by side on a skyline if and only if they have no common factors.

Skylines can also be used to visualize some deeper number theoretical questions. For example, since there are infinitely many primes and primes are sufficiently dense within integers, there are always empty places greater than 1 on the skyline. In particular, the smallest of these is always a prime.

Artistic Viewpoints: Patterns Inside Patterns and Feeling Divisibility

Figure 5: *Towers* 2 *and* 3 *in Divisible Dreams*

Figure 6: *Towers* 5 *and* 6 *in Divisible Dreams*

Figure 7: *Towers* 2, 4 *and* 8 *in Divisible Dreams*

Different visual patterns emerge from different skylines. Inside most skylines one can also find interesting subpatterns. In *Divisible Dreams* we can see the skyline of 2 and 3 and their joint towers on the place

$\mathrm{lcm}(2,3) = 6$ as a repeated pattern (Figure 5). Sometimes two different towers seem to drift apart and then slowly crawl closer again. In *Divisible Dreams* we can see this happening to 5- and 6-towers (Figure 6). They start next to each other, then drift apart and then come close again until they meet in the same place $\mathrm{lcm}(5,6) = 30$. Then the process starts again. Another interesting pattern might be 2-, 4- and 8-towers that form a nice dyadic pattern to which one might wish to add 16-towers and 32-towers and so on (Figure 7).

In *Divisible Dreams* (Figure 1) Lehto has also depicted two other interpretations of *Divisible Skylines*. In the necklace hanging from the tower circle, the integers of a $(3,5,6,10)$-skyline have been translated into circles that resemble pearls or stones. Some of the pearls overlap, but the smaller pearls have been depicted in front of the larger. Behind the necklace a light tapestry pattern can be discerned. It is made of overlapping $(2,3,4,6,8)$-skylines, where circles of radius n are drawn in place of n-towers. These circle skylines are not distinguishable but their outlines echo the interplay between the overlapping different sized circles.

A curious power of *Divisible Skylines* is that as you draw skylines, the divisibility of different integers manifests for you in an embodied way. You can feel the divisibility as towers appear on the skyline. You become conscious of different mathematical phenomena by simply playing with the skylines and you naturally start asking questions about the patterns and shapes that arise. Giving your own interpretations for different skylines, the properties of your chosen integers get personal and multifaceted meanings.

Summary and Conclusions

We present a medium for visualizing number theory. Our method, *Divisible Skylines*, is a mathematically illuminating way of representing least common multiples and divisibility, and we find it visually pleasing.

Our method opens interesting possibilities for mathematics, art and education. We offer a visualizing process that both motivates and deepens mathematical understanding through a creative experience. We feel strongly that this approach would be useful in mathematics education. We are in the process of expanding our project, including developing an open online tool for the use of artists, educators and students alike.

The artwork *Divisible Dreams* offers an example where rigorous mathematical rules and purely artistic visions can support each other in the creative process. Art does not need to merely interpret mathematics and lose its distinct creativity. Nor does mathematics need to serve as a simple tool for artists to use how they want and discard after. Together they can make each other stronger.

Acknowledgments

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