Ceramics Inspired by String Theory

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Abstract

I describe a series of 30 ceramic works that I made to represent my research in mathematical physics. My research is concerned with finding the potential between charged particles in a supersymmetric field theory. After some introduction to the math and physics involved I outline how the calculation is done and the mathematics required for it. I mostly present it through photographs of those ceramic vessels. The forms of the pots is related to the question studied and they are all decorated with details of the calculations.

Some Physics and Mathematics Background

Quarks, QCD and Wilson Loops

One of the most important open questions in mathematical physics is the proof of quark confinement. Quarks, the constituents of protons and neutrons, can be liberated of these composite particles only for very short times before finding other quarks to join in forming new composites. The theory that governs them is known as quantum chromodynamics (QCD), or the theory of strong interactions. Proving a mathematically rigorous version of this statement is one of the Clay Mathematics Institute's Millennium Problems [2].

A common way to characterize confinement is by measuring the potential between particles. In classical electromagnetism (and gravity) the force F at a distance r follows the inverse square law, $F(r) \propto 1/r^2$, and the potential V is (minus) the integral of that, $V(r) \propto 1/r$. In confining theories we expect a force that does not decrease with distance, so $F(r) \sim$ const and $V(r) \propto r$. This means that the energy required to separate particles grows with the distance.

The potential between charged particles is captured by a quantity known as a Wilson loop [13]. Mathematically, this is the holonomy of a vector bundle along a specified path, but it is enough for our purpose to know that Wilson loops are defined for any closed curve C in space-time as quantum operators $W[C]$ and one can calculate their expectation values (quantum averages) $\langle W[C] \rangle$. Physically this captures the effect of a charged particle transversing the path. One often considers the case of a pair of charged particles created in the distant past, separated by a distance r and held fixed for a long time T. We effectively end up with a Wilson loop whose path C is a rectangle of width r and length T (in the time direction). We find then $\langle W[C] \rangle = f(r, T)$ for some function f. In fact, if T is very large, we expect a scaling with T of the form

$$\langle W[C] \rangle \sim \exp(-TV(r)) \,.$$

The argument of the exponent is proportional to the potential $V(r)$ between the particles. Now we see that in the confining case the exponent is proportional to the area enclosed by the curve C: $Tr =$ Area, and indeed we can think of a flux tube stretched between the two particles trying to minimize the area.

$\mathcal{N} = 4$ *Supersymmetric Yang-Mills, String Theory and* AdS/CFT

Most of my research is not on QCD but on another quantum field theory, known as $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM), or maximally supersymmetric Yang-Mills. It is not a theory of our subatomic

quarks, but it has many features in common, including the existence of Wilson loop observables. One striking difference is that this theory is conformal and in particular invariant under rescaling, which precludes confinement, as in this case $V(r) \propto 1/r$, like the Coulomb potential of electromagnetism. The focus of my research [5] is to evaluate the proportionality constant in this relation, which depends on a parameter of the theory, known as the 't Hooft coupling, and denoted by λ.

We would like to understand the theory for all values of λ. Traditionally one studies quantum field theories at weak coupling (small λ) in a Taylor series in terms of integrals represented by Feynman diagrams. A remarkable fact about this particular theory is that the limit of $\lambda \to \infty$ also has a simple description as a string theory. Specifically, it is described by type IIB string theory on a space-time with geometry $AdS_5 \times S^5$ [10].[1] For our purposes all we need to know is that we are dealing with strings and that AdS space, which stands for anti-de Sitter, is a pseudo Riemannian hyperbolic space. In the following I simplify matters and eliminate or ignore the time direction, so we can think of the space on which the string lives as usual hyperbolic space, the realm of the famous Escher drawings [8] (S^5 is the five dimensional sphere, which plays a minor role in the following).

So if we want to evaluate Wilson loops in $\mathcal{N} = 4$ SYM, we either have to do a perturbative calculation, which gives the Taylor expansion around $\lambda = 0$ or some calculation in string theory. For the string calculation, the prescription is to draw the contour C of the Wilson loop on the boundary of hyperbolic space and attach the string to it [11]. Like a soap film, the string tends to shrink and form a minimal area surface. So again we have this picture of a flux tube stretched between the two sides of the Wilson loop contour as in QCD, but now they are allowed to extend into the 5th dimension of our hyperbolic space, and not only remain on the boundary.[2]

The simplest non-trivial Wilson loop, that with a circular contour, $W[\circ]$, can be evaluated exactly [9, 6, 12]. In that case one can evaluate the Feynman diagrams (or use fixed point and index theorems) to find the full Taylor expansion around $\lambda = 0$ summing up to the function

$$\langle W[\circ] \rangle = \frac{2}{\sqrt{\lambda}} I_1 \left(\sqrt{\lambda} \right),$$

where I_1 is a modified Bessel function. The asymptotic expansion of this function at $\lambda \to \infty$ gives

$$\langle W[\circ] \rangle \to \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}},$$

which agrees with calculations from string theory in AdS space.
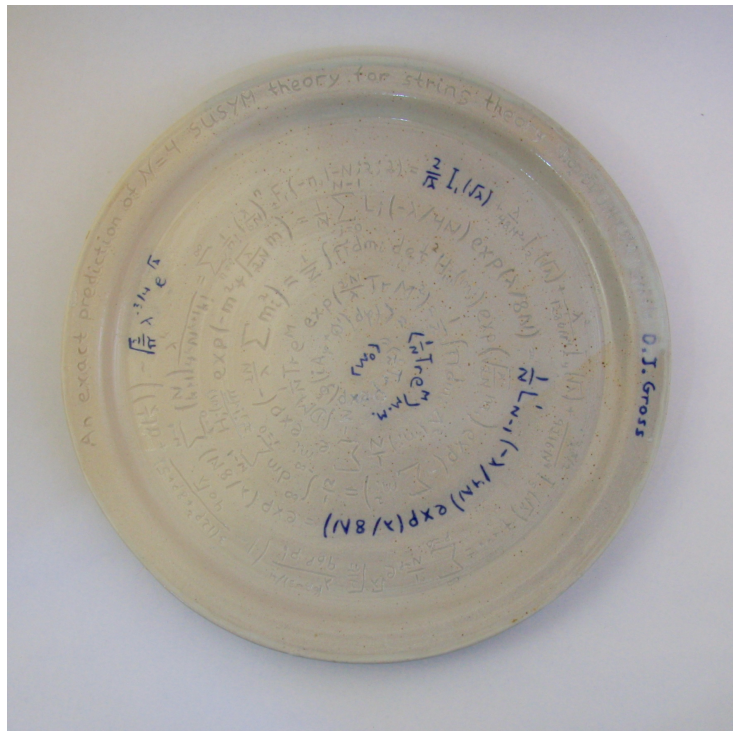
Figure 1: *Circle-6, incised and inlaid stoneware, clear glaze. 3 × 32 × 32cm.*
A plate with the calculation of the circular Wilson loop.

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[1]This statement is not prooven in the mathematical sense. We don't have a fully rigorous definition of either side of this duality, but in any circumstance where we can formulate this correspondence exactly and check it, it is verified to hold.

[2]Mathematically we are dealing with the hyperbolic version of the Plateau problem.

Artistic practice

For each of my research projects I choose a form to represent the topic of study, like the circular plate with spiral writing in Figure 1 for the circular Wilson loop. I then make a series of pieces highlighting different stages and aspects of the project. I endeavor to start making the ceramics in parallel to the research, so early works include rough ideas and draft calculations. I render them in stoneware, often with rough writing and sometimes in the later firing stages obscure the writing with the glaze (see *e.g.*, Figure 9 below).

After the research is complete and published, I start using porcelain and place more refined presentations of my final results on it. I employ finer decoration techniques, like inlay and carving and also apply precious metal lusters. I also write its arXiv number, or its web address, as this is another representation of my results.

I now turn to present my research with V. Forini published in [5] and captured in a series of 30 pots with the names Cusp-n for $1 \leq n \leq 30$. In the discussion below I explain the physics and math and illustrate it with the ceramic creations. I also try to outline some of the design elements in the choice of form and decorations related to the math and academic practice. For example, the reference list on which our paper is based is imprinted near the base of Cusp-20 in Figure 10.

Generalized Quark-Antiquark Potential

We want to calculate the analog of the potential between quarks, so the Wilson loop corresponding to two parallel lines $W[\|]$, which turns out to be much more complicated than the circle. The trick used in [5] is to add an extra parameter to our story and instead of parallel lines, consider lines at arbitrary angles $W[\angle]$. I take the straight line to be the angle $\phi = 0$ and the parallel lines to be $\phi = \pi$. Now recall that the theory is conformal and that under conformal transformations lines go to circles,[3] so the case of $\phi = 0$ is the same as the circle described above. In all other cases, each of the rays emanating from the angle gets mapped to an arc, and the pair forms the shapes on the right of Figure 2.
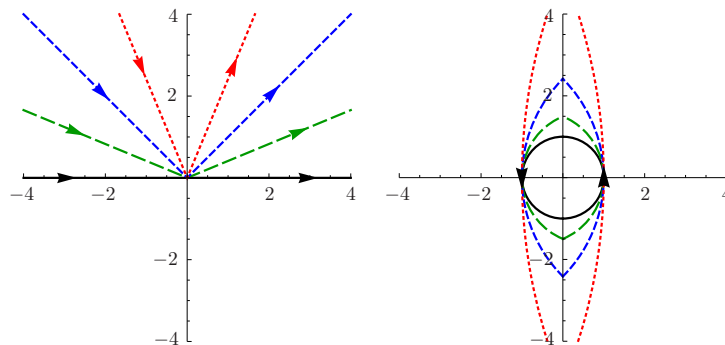

Figure 2: *Pairs of rays at angles $\phi = 0, \pi/4, \pi/2, 3\pi/4$ get mapped by different conformal transformations to pairs of arcs, interpolating between the circle and a pair of antiparallel lines.*

Note that in the picture on the right, I kept the distance between the middle of the arcs fixed, which is again possible by a conformal rescaling, so in the limit $\phi \to \pi$, we do get two parallel lines at finite distance.

There is an extra parameter one can introduce into the problem (related to the S^5 in the string picture), which does not complicate the story by much. It is another angle that I label θ. So finally we are looking to calculate a Wilson loop which depends on ϕ and θ in addition to the parameter λ. The original question, of $V(r)$ can be found from the residue of the pole at $\phi \to \pi$ (with $\theta = 0$).

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[3]The conformal group in four dimensions is $SO(1,5)$ or $SO(2,4)$. It is similar to the $SL(2,\mathbb{C})$ Möbious transformations of the complex plane.
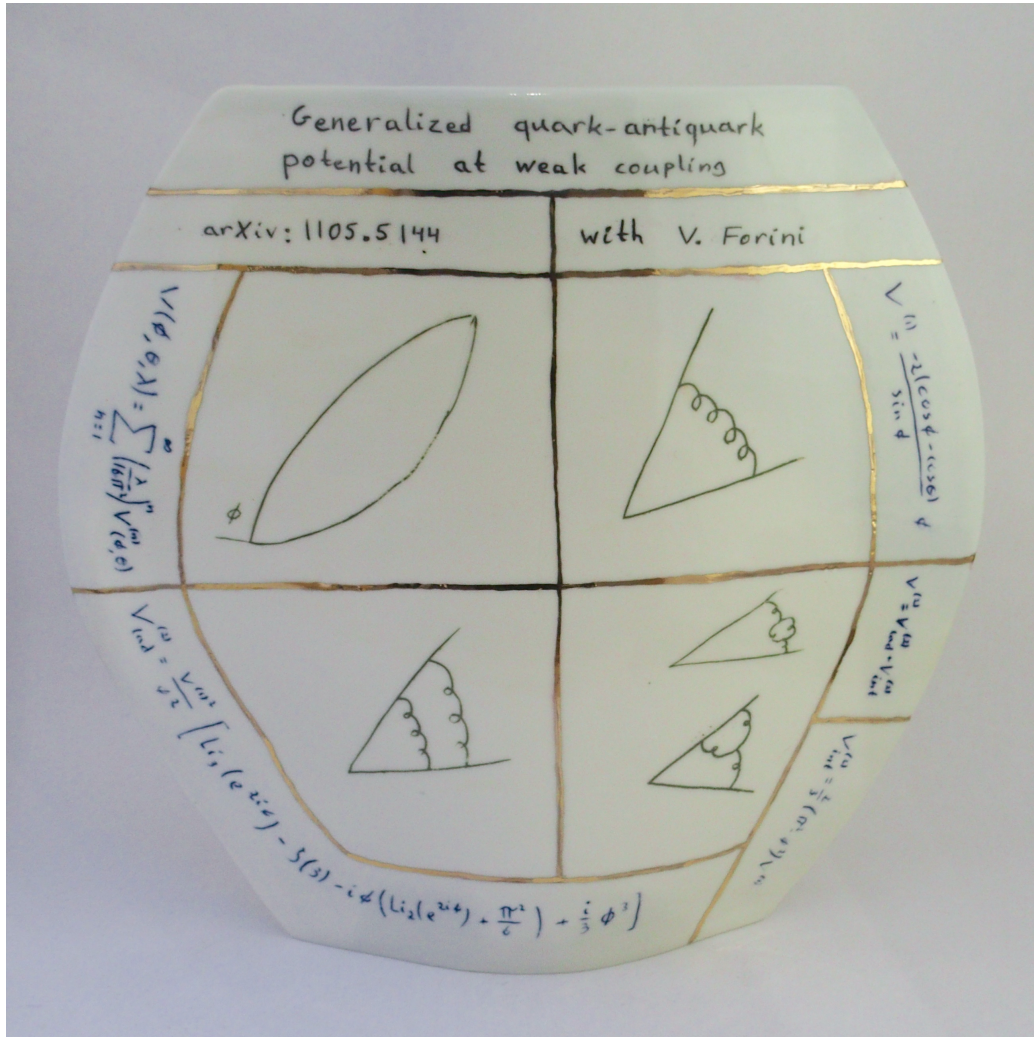
Figure 3: *Cusp-21, inlaid porcelain, clear glaze and gold luster.* $30 \times 32 \times 6cm.$
The path of the Wilson loop as in Figure 2 (top left), a one-loop Feynman diagram (top right) and two-loop Feynman diagrams (bottom). The results of those integrals are written on the perimeter.

In Figure 3 you see the small-λ calculation summarized in porcelain inlaid with cobalt-blue, chrome-green and black slips and decorated with real gold luster lines. The squiggly lines represent gluons, the charge carriers of the strong interactions (analogs of photos, the quanta of light) which get exchanged between the legs of the angled Wilson loop.


Figure 4: *Cusp-30 top view, porcelain, oxblood glaze and gold luster.* $12 \times 10 \times 3cm.$

These figures, known as Feynman diagrams, represent particular integrals via the "Feynman rules". Evaluating the integrals, as always, can be an easy or a hard endeavor. We evaluated all the integrals contributing up to

$O(\lambda^2)$ and the results of the integrals are summarized around the pot.

Figure 4 shows a top view of another pot. The design is such that the cross-section matches the curves in Figure 2 (and top left image in Figure 3), where the angle between the two slabs from which the pot is made represents the angle ϕ in the calculation.

In Figure 5 we see the detail from another pot, where the result of the Feynman diagram integral proportional to λ in the Taylor expansion is presented. The number 7.70 is calculated by plugging in $\phi \sim 2.13$ and $\theta = 0$. This value of ϕ is the angle that I measured on this pot (and $\theta = 0$ is the more physically interesting case).
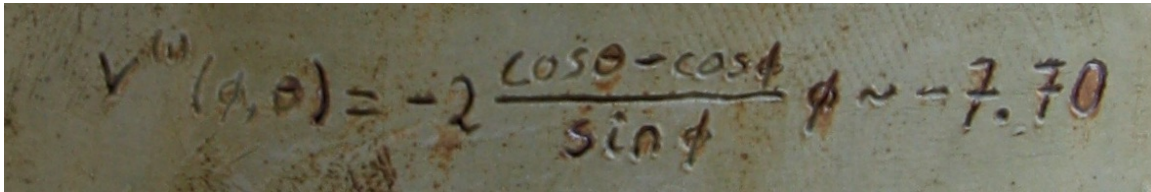
Figure 5: *Cusp-1 detail, incised stoneware, red iron oxide and celadon glaze.* $28 \times 23 \times 7cm$. *The result of the one-loop Feynman diagram calculation.*

Another motivation for the shape of the pots is that I dedicate each side to one approach to the problem, the second side represents the string theory calculation, see Figure 6.

Figure 6: *Cusp-29 AdS side, porcelain, oxblood glaze and gold luster.* $28 \times 30 \times 9cm$. *The color pattern is the random result of partial oxidation in the kiln.*

The calculation of the minimal surface ending on an angle at the boundary of *AdS* space was originally done in my very first paper on string theory during my PhD [7]. In [5] we generalize that result.

In Figure 7 to the right (and also the top of Figure 6) is a graphical representation of the minimal surface. Here I tried to represent the Poincaré patch of *AdS* space, or the upper half space model of hyperbolic space. Since the angle is in the plane $\mathbb{R}^2$, once we add the extra direction we need to consider only AdS_3 (or $\mathbb{H}_3$), which is not five dimensional, but still three dimensions are hard to draw. And since I already used up my three dimensions to make the pots with the cross-section based on the angles, I now had to resort to drawing in perspective.

In this figure the boundary of *AdS* space is at the bottom, where you see an angle. The arch extending above it is the string. The formulas describing this embedding are given in parametric form by elliptic integrals. Those are generalizations of (inverse) trigonometric functions, which my collaborator V. Forini and I had the dubious pleasure of studying (see *e.g.* [1]).

Figure 7: *Cusp-16 detail, incised stoneware with shino glaze.* $29 \times 26 \times 7cm.$

After finding the solution for the shape of the string, one can evaluate its area, which diverges, since the distance to the boundary of hyperbolic space is infinite. But there is a way of removing this infinity to get a finite quantity (known as "renormalization"). The formula for the area is as also written on Cusp-1, see Figure 8.
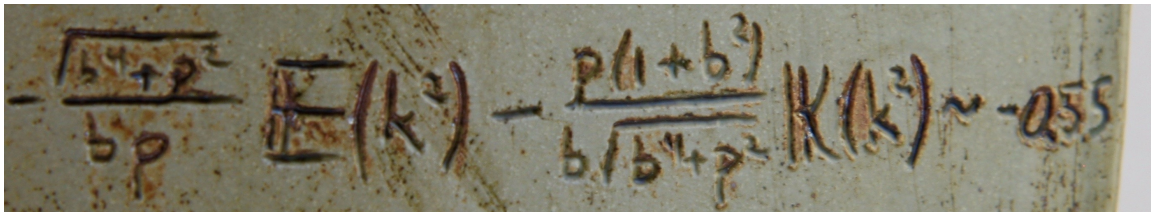
Figure 8: *Cusp-1 detail, incised stoneware, red iron oxide and celadon glaze.* $28 \times 23 \times 7cm.$
The result of the cassical string calculation.

Here $\mathbb{K}$ and $\mathbb{E}$ are complete elliptic integrals of the first and second kind. k is their modulus and there are equations relating k, b and p to the angles ϕ and θ, which are found elsewhere on the pot.

There is one more useful conformal transformation, which instead of mapping $\mathbb{R}^4$ to itself, maps it to $S^3 \times \mathbb{R}$. Starting with polar coordinates, we express the radius as $r = e^\tau$. Now the two rays emanating from

the origin are mapped to two lines along τ at fixed points on S^3 (separated by an angle ϕ), so the "generalized quark-antiquark potential" corresponds to a pair of static particles on the sphere. When we go to the string picture, we fill the sphere to a ball with the hyperbolic metric. You can see in the middle right of Figure 9 my attempts to draw the surface ending along two lines on the cylinder.

Figure 9: *Cusp-2, incised stoneware, red iron oxide, shino and glue glazes.* $34 \times 21 \times 6cm$.

In our paper we also calculate the first correction to the classical string which arises from its fluctuations. This requires evaluating determinants of differential operators, the Lamé equation and some more advanced math. I outline that calculations on some of the pots, for example the bottom of Cusp-29 in Figure 6.

Summary and Conclusions

I have tried to explain here my research on the generalized quark-antiquark potential in $\mathcal{N} = 4$ SYM and my practice of realizing my research in ceramics. The paper these pots are based on led to a pair of works that used tools of integrable systems to completely reformulate the problem of the quark-antiquark potential [4, 3] and some results at all values of λ, but that will be the topic of a different series of ceramic vessels.

The experience of inscribing my calculations on ceramics has been exhilarating. I noticed that many people, who would otherwise find mathematics intimidating and formulas repelling, were instead attracted to it. Viewed as hieroglyphs or cuneiforms, the observer is apparently not confronted with their lack of understanding, but is drawn to it as a foreign mysterious writing system. It also allows me to reexamine my scientific research and find ways of representing it in 3d forms with 2d decorations. My collaborators and colleagues have been intrigued and amused by this way of expressing scientific research and it opened for me doors to museums and galleries, which would otherwise not be so welcoming to a traditional potter.

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Figure 10: *Cusp-20 detail, impressed stoneware with celadon glaze.* $36 \times 30 \times 10$ *cm. The references of our paper [5] are impressed on the base of this pot.*