# Mapping Diagrams and a New Visualization of Complex Functions with GeoGebra

Martin Flashman[1]

[1]Mathematics Department, Humboldt State University, Arcata, CA; flashman@humboldt.edu

## Abstract

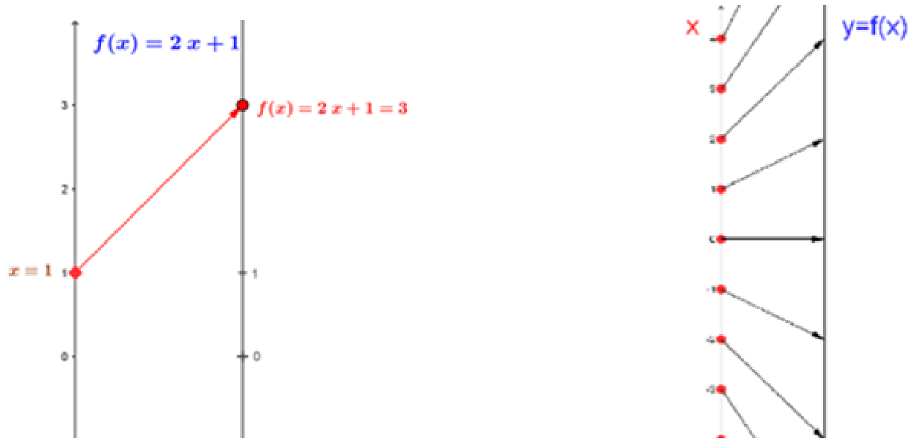
In the 21st century the interactions of technology, mathematics, and the visual arts have grown rapidly, developing new and exciting connections. One such nexus has been through visualizations created with technology related to the geometry and analysis of complex functions. It is sometimes said that the goal of complex analysis is to generalize the calculus of real variables to a calculus for complex variables The real calculus emphasizes the graph of the function as the main visualization, but a graph for complex functions needs a four dimensional Euclidean space. Not an easy task. I introduce new dynamic and interactive mapping diagrams created with GeoGebra that enhance the study of complex functions without four dimensions, while providing new visualization tools for the artist.

## Introduction

The work of a visual artist creates and transforms understanding by capturing the sometimes abstract nature of reality in a picture, sculpture or video. In the 21st century the interactions of technology, mathematics, and the visual arts have grown rapidly developing new and exciting connections. One such nexus has been through visualizations created with technology related to the analysis of real and complex functions, including fractal geometry and the symmetries of planar and spatial transformations. Introduced in 2015, GeoGebra version 5 added a dynamic and interactive 3 dimensional workspace along with an ample supply of tools for working with complex numbers and functions. These improvements came at just the right time for me as I was teaching an undergraduate course in complex analysis at Humboldt State University. Over the next three years, I developed the mapping diagram visualizations for complex functions presented in this paper. These offer the visual artist both new tools and perspectives for illuminating the geometry and beauty of complex functions.

**A real mapping diagram** (RMD) is an alternative to the Cartesian graph for visually understanding real valued functions that provides clarity for several procedures and concepts not handled easily with a graph. The real number argument, $x$, of a real valued function, $f$, is visualized as a position on a number line, and the corresponding real number output, $f(x)$, is visualized on a parallel number line. In this way, the first line represents the domain, and the second line represents the codomain of the function. An arrow from the input to the output is used to visualize the function relation between the two real numbers. See Figure 1(a). A RMD can display many arrows conveying information comparable to a table of values for the function. See the mapping diagram in the Figure 1(b).

The use of RMDs to understand functions can be identified at least as far back as Napier's introduction of logarithms, [8]. In the last 70 years, RMDs have been used often for visualizing concepts in a calculus context, *e.g.*, [11], [2], [13], [5]. For over 30 years, I have been producing materials using RMDs to support instruction. See [3] and [4]. Those materials provide a thematic approach to algebra and calculus, making consistent use of RMDs to visualize functions and develop a variety of concepts. The geometry of RMDs is also related to point line configurations in projective geometry, see [7], and string art, see [10].




**(a)** **(b)**

**Figure 1:** *Real mapping diagram examples: (a) A real mapping diagram for a linear function for a single input, (b) A real mapping diagram for a linear function on a sampling of inputs.*

A **complex mapping diagram** (CMD) can be created for functions of complex numbers by replacing the two axes of the real mapping diagram with two planes, visualizing the complex function's domain and co-domain. CMDs also have a long history in the visualization of complex analysis, *e.g.*, [1], which has been rejuvenated recently in [9] and [12]. In the past, these diagrams followed the convention of visualizing the domain and codomain as separate bounded regions in the same plane as the printed page with some limited number of arrows between corresponding points or corresponding curves in the two regions. See Figure 2.

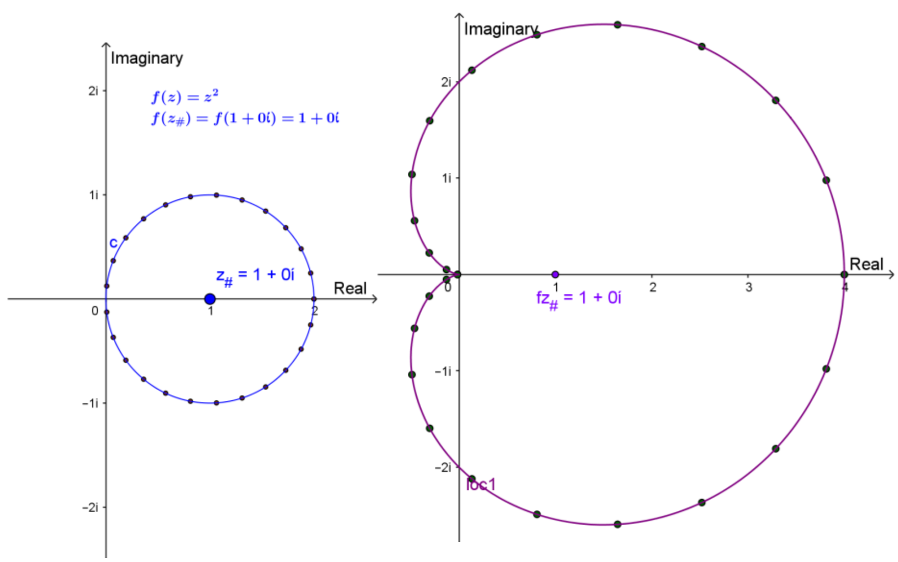
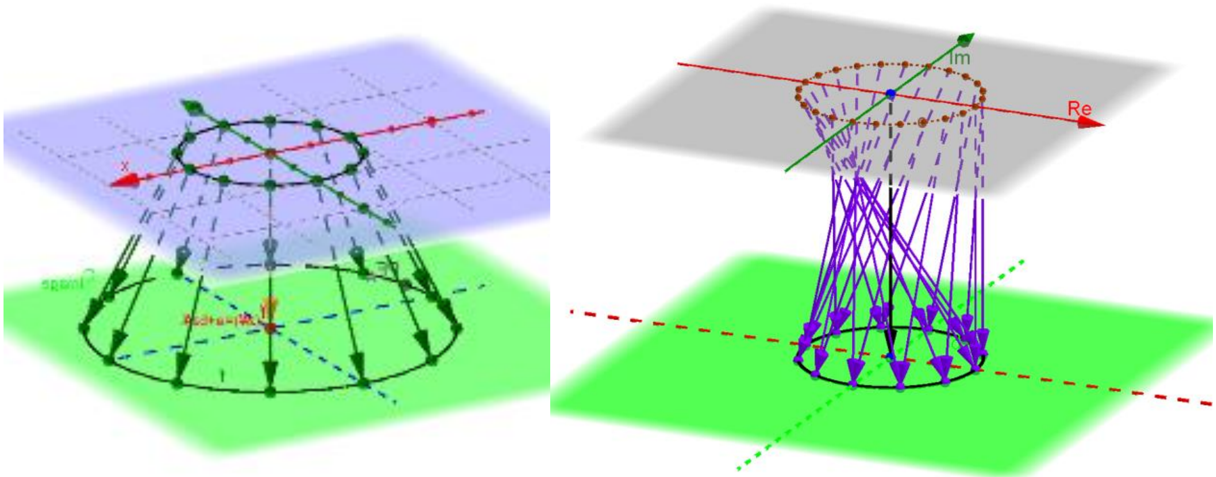



**Figure 2:** *Coplanar Complex Mapping Diagram for $f(z) = z^2$ applied to the circle* $|z - 1| = 1$

With the introduction of 3 dimensional tools in GeoGebra 5.0, I extended the use of RMDs to complex analysis by using parallel complex planes in a three dimensional context for the domain and codomain. As with a RMD, an arrow from an input, $z$, to the corresponding output, $f(z)$, is used to visualize the function relation between the pair of complex numbers. CMDs provide additional geometric insights to key concepts, tools, and applications of visual complex analysis. Figure 3(a) shows a parallel plane CMD for a complex linear function on the circle $|z| = 1$. Figure 3(b) shows a CMD for the complex quadratic function $f(z) = z^2$ for the circle |z|=1, presenting a double covering of the unit circle. A similar treatment of CMDs was introduced independently by Yoav Yaari in [7].




**(b)** **(b)**

**Figure 3:** *Complex mapping diagram examples: (a) A complex mapping diagram for a linear function on a sampling of points chosen on a circle,* $|z| = 1$*. (b) A complex mapping diagram for* $f(z) = z^2$ *on a sampling of points chosen on a circle,* $|z| = 1$.

A dynamic visualization of a real or complex mapping diagram allows for direct user control of a slider for a real variable or the point in the real or complex domain to show how the argument effects the value of the function. Other interactive features can change the domain curve being selected in a CMD. These features are all available with GeoGebra 5.0. I have linked examples of dynamic interactive diagrams in the GeoGebra book related to this paper, [5]. All of these provide new visualizations and tools for a visual artist to explore connections to old and new 3 dimensional line figures and surfaces.

This paper presents examples of 3 dimensional CMDs with corresponding analogous RMDs as visualizations illustrating interesting connections between geometry and complex functions, including treatment of linear and quadratic functions as well as linear fractional (Möbius) transformations. Since space is limited, it is left to the reader to discover further details of CMDs that can add meaning and new visualizations to other parts of complex (and real) analysis. See [5].

## Complex Functions with Mapping Diagrams

### *Linear Functions*

For a real linear function, such as $f(x) = 2x + 1$, it is not hard to show that the family of lines determined by the arrows in its RMD all pass through a common focus point. Thus a RMD visualizes a real linear function as a dilation with a focus point, as in Figure 4. The position of the focus point (possibly at “infinity”) is determined by the relative position and scales of the parallel number lines.

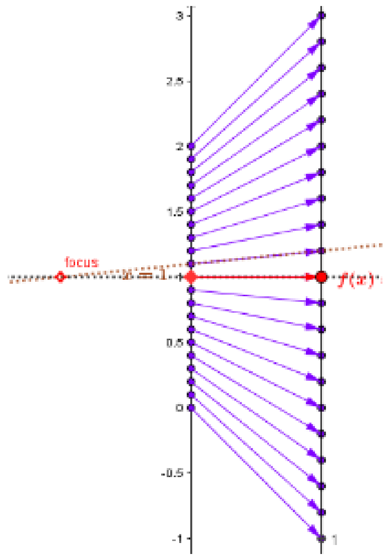
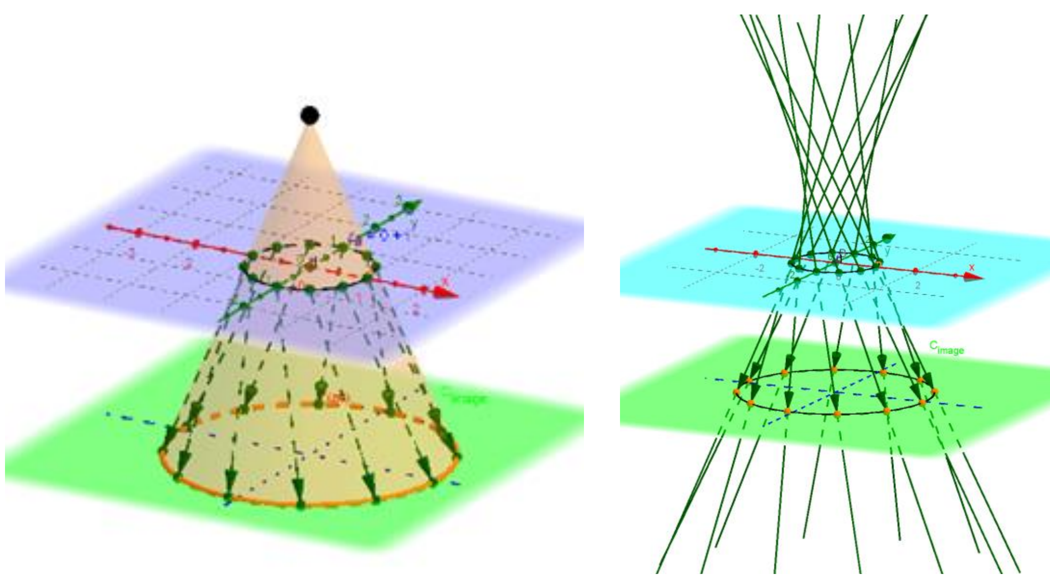



**Figure 4:** *Real Mapping Diagrams for* $f(x) = 2x + 1$; $f(0) = 1, f(1) = 3$.

The CMD for a simple linear function of complex multiplication, for example $f(z) = (2 + i)z$, also arises from a dilation from a single focus point – the vertex of a cone determined by multiplication by the real number, $\left||2 + i|\right| = \sqrt{2^2 + 1^2} = \sqrt{5}$, followed by a twist-rotation resulting from the angle determined by the complex number multiplier, $2 + i$. In this example the lines determined by the arrows for a circle in the CMD domain all lie on a ruled hyperboloid of one sheet passing through the domain circle and its image in the codomain. See Figure 5.



**(a)** **(b)**

**Figure 5:** *Complex Mapping Diagrams for* $f(z) = (2 + i)z$ *for the circle* $|z| = 1$, *(a) Dilation followed by rotation; (b) Hyperboloid of one sheet.*

***Quadratic Functions***

For the core real quadratic function, $f(x) = x^2$, the family of lines determined by the arrows in its RMD envelop a hyperbolic curve. See Figure 6.

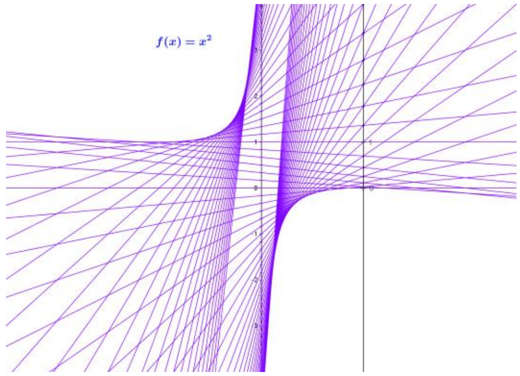
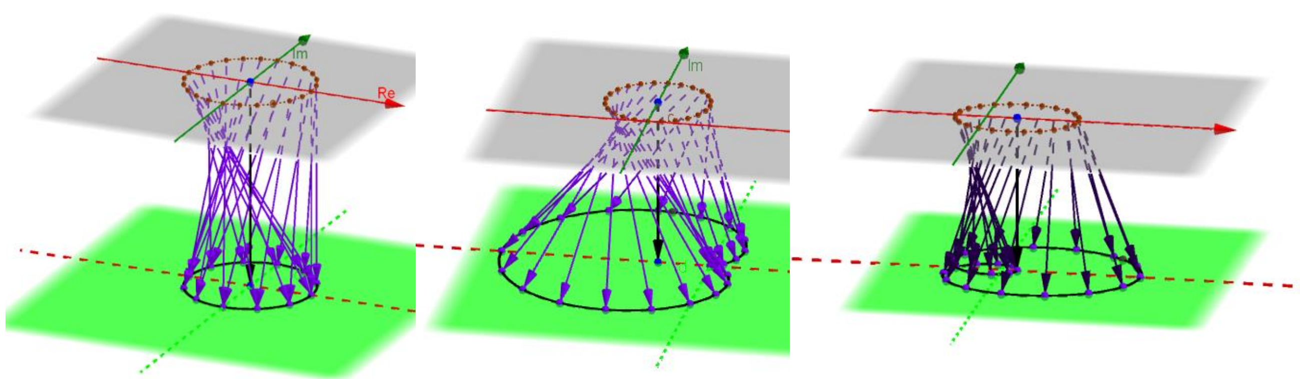



**Figure 6:** *Real Mapping Diagram for* $f(x) = x^2$

The CMD for the core complex quadratic function, $f(z) = z^2$, provides a variety of visually interesting spatial configurations determined by the arrows for a circle in the domain plane while the lines determined by the arrows for a circle in the CMD domain lie on a ruled surface. For example, using the unit circle centered at 0, $|z| = 1$, the arrows form a double covering of the unit circle in the codomain, an interlaced pattern lying on a self-intersecting surface bounded by two circles. See Figure 7(a). For the unit circle centered at $z = i$, $|z - i| = 1$, the arrows connect the circle to a heart shaped image in the codomain passing through 0. See Figure 7(b). As a third example, for the unit circle centered at $z = \frac{1}{2}$, $\left|z - \frac{1}{2}\right| = 1$, the arrows connect the circle in the domain plane to a self-intersecting closed Limaçon curve in the codomain plane. See Figure 7(c). In GeoGebra, these three figures can be linked dynamically by "grabbing" the center of the unit circle and moving it from $z = 0$ to $z = i$, and then to $z = \frac{1}{2}$ in the domain plane. See [5].

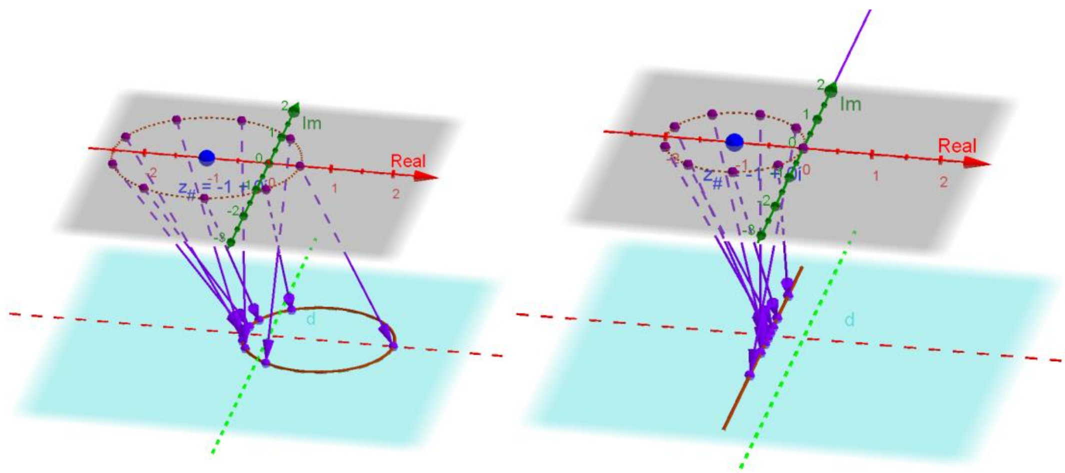



(a) (b) (c)

**Figure 7:** *Complex Mapping Diagram for* $f(z) = z^2$
*applied to the circle:* $(a)$ $|z| = 1$, $(b)$ $|z - i| = 1$, $(c)$ $\left|z - \frac{1}{2}\right| = 1$.

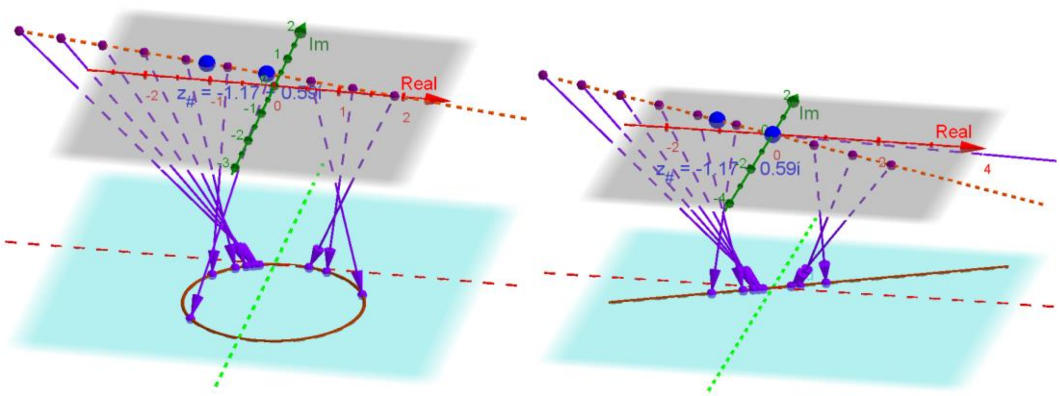
***Complex Linear Fractional (Möbius) Transformations***

The complex linear fractional (Möbius) transformations are functions of the form $f(z), = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Of great interest for their geometric and analytic properties, these transformations leave invariant the family of circles and lines in the complex plane. This visual feature is evident even in the simplest of these functions, $f(z) = \frac{1}{z}$: See Figures 8 and 9. Note that in each of the figures, when the circle or line in the domain passes through the point $z = 0$ the image in the codomain is a line. In GeoGebra, these figures can be linked dynamically by varying the radii of the domain circle centered at $z = -1$ from 1.5 to 1, or moving the domain line so the imaginary axis intercept changes from $z = \ i$ to $\ z = 0$. See [5].




**(a)** **(b)**

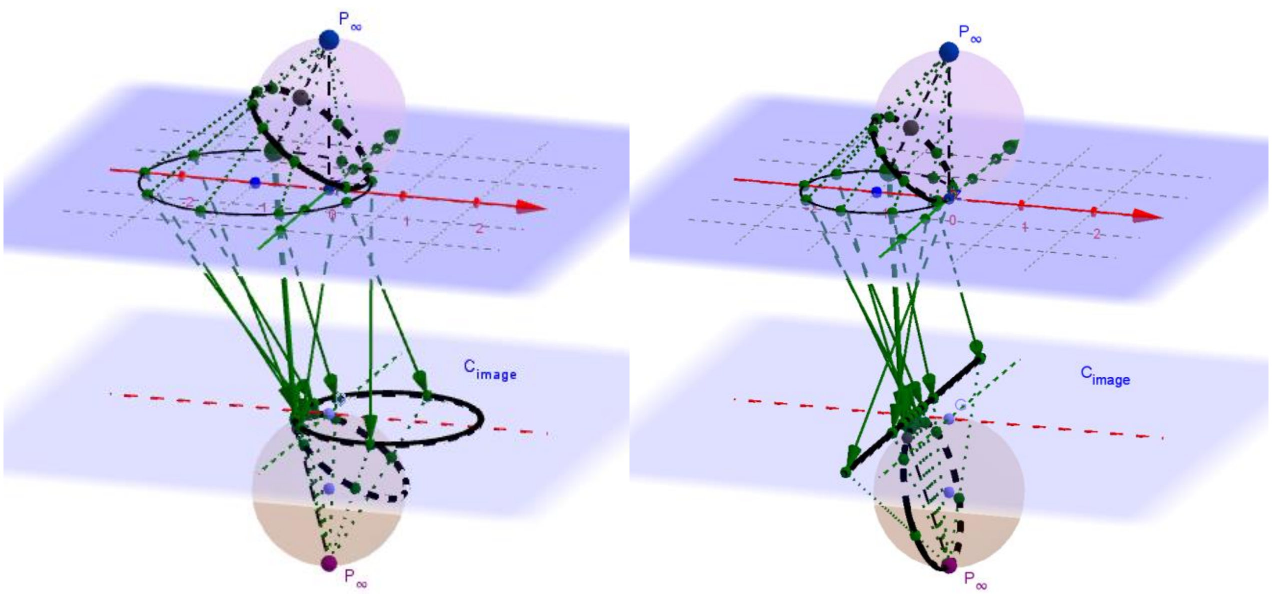
**Figure 8:** *CMD for* $f(z) = \frac{1}{z}$*:* $(a)\ \ |z+1| = 1.5,\ \ (b)\ |z+1| = 1$




(a) (b)

**Figure 9:** *CMD for* $f(z) = \frac{1}{z}$*: Line through: (a)* $z = i$*, (b)* $z = 0$*,*

Mapping diagrams also help connect complex linear fractional (Möbius) transformations to related spherical geometry transformations in a more intricate fashion as compositions. The diagram for $f(z) = \frac{1}{z}$ starts by projecting the Riemann sphere from the pole that visualizes the complex point at infinity, $P_\infty$, to the complex plane, then applies the linear fractional transformation between the two complex planes, followed by the projection from the complex plane back to the Riemann sphere. These diagrams transform the family of circles on the initial sphere to the family of circles on the final sphere. Note that a line in the complex plane corresponds to a circle on the Riemann sphere passing through the pole that represents the complex point at infinity $P_\infty$. See Figure 10.




(a) (b)

**Figure 10:** *CMD for* $f(z) = \frac{1}{z}$ *with Riemann Spheres: (a)* $|z + 1| = 1.5$, *(b)* $|z + 1| = 1$,

## Conclusion: Not the End. Only a Beginning

Complex mapping diagrams created with the technology available with GeoGebra provide much beauty and power in visualizing complex functions while allowing artists to envision new objects based on these function. Some of the benefits of the use of these tools are evident in the figures I've included here. Other benefits to educators extend to the visualization of the analysis and geometry of complex analysis, including solving equations, differentiation, integration, and infinite series. Even more remains to be discovered in their dynamic aspects revealed using GeoGebra [5].

## Acknowledgements

I would like to thank Sally Scott for her assistance in clarifying some of the language and direction in this paper. I would also like to thank the referees for their helpful suggestions on the paper's focus. Any errors or confusion in the paper are my own responsibility.

## References

[1] L. Ahlfors. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*. 1st Edition, McGraw Hill, 1953.

[2] T. Brieske. "Mapping Diagrams, Continuous Functions and Derivatives." *College Mathematics Journal*, *9*, 1978, pp. 67-72.

[3] M. Flashman. Mapping Diagrams from A(lgebra) B(asics) to C(alculus) and D(ifferential) E(quation)s. A Reference and Resource Book on Function Visualizations Using Mapping Diagrams. https://flashman.neocities.org/MD/MDbook_%20homepage.html, 2013.

[4] M. Flashman. *Sensible Calculus Program*. https://flashman.neocities.org/flashman/senscalc.Core.html, 2014.

[5] M. Flashman. *Mapping Diagrams and Visualization of Complex Function*. GeoGebra book, https://ggbm.at/gutmhcp8, 2019.

[6] L. Gonick. *The Cartoon Guide to Calculus*. (Cartoon Guides) Paperback, HarperCollins, 2012.

[7] A.Inselberg. *Parallel Coordinates: Visual Multidimensional Geometry and Its Applications*. Springer, 2009.

[8] J. Napier. Mirifici logarithmorum canonis description. 1614.

[9] T. Needham. *Visual Complex Analysis*. Oxford: Clarendon Press, 1997.

[10] Pohl, V. *How to Enrich Geometry Using String Designs*. National Council of Teachers of Mathematics, Reston, VA, 1986.

[11] M. Spivak. *Calculus.* W.A. Benjamin Inc., 1967.

[12] I. Stewart & D. Tall. *Complex Analysis.* 2nd Edition, Cambridge University Press, 2018.

[13] H. Swann & J. Johnson. *Prof. E. McSquared's Calculus Primer*. Expanded Intergalactic Version, Janson Publications, 1989.