Interlacing Mathematics and Art: Hands-on Non-Euclidean Geometry

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Abstract

Elliot Eisner (1933-2014), a pioneer in arts education, suggested that an artistic approach to education could improve its quality and lead to a new vision for teaching and learning. This is true for any subject, including mathematics. Geometry related topics make a perfect setting for a deeper contribution of art to education. One such topic is the study of non-Euclidean geometry. In this paper, we present some results of a professional development course for mathematics teachers where the participants studied non-Euclidean geometry concepts. The participants created ceramic pieces using the Poincaré disk and styrofoam balls covered with fabrics using spherical geometry.

Introduction

Mathematics classes in schools are usually described by students as being uninteresting and boring. Many researchers have tried to provide solutions and suggestions to deal with this problem. Elliot Eisner, a pioneer in arts education, believed that a conception of practice rooted in the arts should contribute to the improvement of both the means and the ends of education [3].

Engaging students in the classroom is a fundamental task for an effective and fulfilling learning experience [5]. In the arts, engagement tends to be secured by the aesthetic satisfactions obtained from the work itself [3]. The work being created presents natural challenges which are related to some of these satisfactions:

> Materials resist the maker; they have to be crafted and this requires an intense focus on the modulation of forms as they emerge in a material being processed. This focus is so intense that all sense of time is lost. The work and the worker become one (p.9).

Eisner [3] believes that the feeling of vitality and the surge of emotion we feel when experiencing an art can be embedded in the ideas teachers explore with students and in the appetite for learning that teachers stimulate: "In the long run, these are the satisfactions that matter most because they are the only ones that ensure that what we teach students will want to pursue voluntarily after the artificial incentives so ubiquitous in our schools are long forgotten" (p.9).

Geometry related topics make a perfect setting for a deeper contribution of art to education and allow for a complete symbiosis between the teaching of mathematics and an artistic education. One such topic is the study of non-Euclidean geometry which is now briefly addressed in middle school mathematics curriculum in Portugal [1].

Today's fast changing pace, in combination with the complexity of mathematics teaching practices, place extraordinary demands on practicing teachers [5]. Professional development is an important means to help teachers find ways to meet these demands. For almost thirty years, the mathematics education community has made a continuous effort to change teaching practices towards being more cognitively demanding, conceptually oriented and student centred [6].

In past years, a large amount of research has been done on the professional development of teachers. Borko, Jacobs and Koellner [2] highlight the recent shifts in professional development methodologies. These shifts are somewhat related to recent shifts in the prominence of ideas about the nature of cognition, learning and teaching. As the authors refer, professional development should move away from a rigidly structured in-service training model towards approaches grounded in classroom practice and involving the formation of professional learning communities. Indeed, as Higgings and Parsons [7] wrote, "professional learning opportunities for teachers need to be situated in the teacher's context of practice and relevant to the teaching and learning needs of teachers and students". These learning opportunities should focus on the real context of the classroom and be integrated into the teacher's everyday work.

In this paper, we present some results of a professional development course for mathematics teachers where the participants were introduced to basic non-Euclidean geometry concepts and created ceramic pieces using (mostly) the Poincaré disk, a model for hyperbolic geometry. Some participants also applied spherical geometry when covering Styrofoam balls with fabrics.

Non-Euclidean Geometry

The first formal theory of geometry was given by the ancient Greek mathematician Euclid in his treatise on the Elements. It became an overwhelming influence in the development of mathematics and particularly of geometry. His treatment has been refined into what is nowadays known as an axiomatic system. A postulate or axiom states a rigorous logical relationship between undefined terms or "primitives", which are just placeholders for whatever the system represents. A small number of such axioms are stated, and from these the theory is rigorously derived [9]. Euclid put forward five postulates, which remained unchallenged for over a thousand years. His fifth postulate concerned parallel lines. The discovery that it could be substituted by other alternatives gave rise to what are known as the non-Euclidean geometries.

In plane geometry, the three basic undefined terms are *point*, *line* and *lie on* and one of the most important definitions is parallelism: *two lines are said to be parallel if they do not have common points,* see [9] and [4]. Recall that, the Euclidean Parallel Postulate states that *for every line l and every point P that does not lie on l, there exists a unique line m such that P lies on m and l and m are parallel* [9].

One parallel postulate that can replace the Euclidean one is the Elliptic Parallel Postulate that states that *for every line l and every point P that does not lie on l, there is no line m such that P lies on m and m is parallel to l* [9]. In some axiomatic formulations of elliptic geometry, an axiomatic "point" corresponds to a pair of diametrically opposed or antipodal geometrical points on the surface. In spherical elliptic geometry, a line is a "great circle". Two such lines meet at two antipodal points. In this model there are no parallel lines (for more information on elliptic geometry see for instance [8]).

In hyperbolic geometry, the Euclidean Parallel Postulate is replaced by the Hyperbolic one that states that *for every line l and every point P that does not lie on l, there are at least two lines m an n such that P lies on both m and n and both are parallel to l* [9].

The Poincaré disk is a model for hyperbolic geometry. In this model, points are the points inside the unit disk and lines are arcs of circles orthogonal to the boundary of the disk, obtained by intersecting these circles with the disk. Lines are also diameters of the disk which can be seen as special cases of arcs with infinite radius, passing through the centre of the disk. It should be noted that the points lying on the limiting unit circle do not belong to the model, these points are usually called ideal points. In this geometry, it is possible to prove that given a line *l* and a point *P*, which does not belong to the line, there exists an infinite number of lines passing through *P* that are parallel to *l,* see for instance [9, p. 331]. In this model, the concept of a polygon is similar to the traditional one (in Euclidean Geometry), where the sides are segments of lines (as described above) meeting at certain points inside the disk. When those meeting points (vertices) are at the border of the disk the polygon is called ideal.

A prominent contribution to the exploration of artistic tessellations was given by M.C. Escher, a Dutch graphic artist who lived from 1898 to 1972. Despite not being a mathematician, his work is strongly influenced by mathematical concepts and Escher made his own mathematical incursions in order to achieve his artistic purposes. Some of his works use hyperbolic geometry, in particular, by tiling the Poincaré disk, using figurative tiles in a similar (but adapted) way as he did in his wallpaper designs contained in his notebooks. Examples of such art works are: Circle Limit with Butterflies (1950), Circle Limit I (1958), Circle Limit III (1959) and Circle Limit IV (1960) (see https://www.wikiart.org/en/m-c-escher/all-works#!#filterName:Series_circle-limit,resultType:masonry). In other works M.C. Escher was inspired by the geometry of the sphere, a model for elliptic geometry (see for instance http://mathstat.slu.edu/escher/index.php/Concentric_Rinds and http://mathstat.slu.edu/escher/index.php/Sphere_with_ Reptiles). The work of Escher served as an initial motivation for the course design and to motivate the participants to engage in the proposed activities.

Description of the Course

The professional development course addressed in this paper is part of a set of professional development courses for mathematics teachers promoted over the last several years following the ideas presented in the Introduction of this paper. In these courses we try to integrate the activities into the teachers' everyday work, while at the same time provide an opportunity to both learn and have a fulfilling and gratifying experience. Following the ideas presented by Borko et al. [2], these courses are intended to be "opportunities grounded in a conception of learning to teach as a lifelong endeavour", that should be both pleasurable and rewarding. At the same time, we try to move towards Eisner's conception of a practice rooted in the arts. Teachers take on the role of an artist: they are given time to explore, to create and to surprise themselves.

The professional development course in focus was titled "Geometrias a torto e a direito" (Geometries in many ways) and took place in the University of Aveiro, in 2018, from January 5th to March 23rd. Like all professional development courses for Portuguese teachers, it was acknowledged by the national scientific and pedagogical council for teacher's professional development (Conselho Científico-Pedagógico da Formação Contínua), and was registered with the number CCPFC/ACC - 92641/17. This course consisted of 25 hours of contact with all participants (plus 25 hours of individual work at home or at school) and had the collaboration of a ceramicist who taught part of the course. The course had 20 participating teachers who taught mathematics from grades 1 to 12. Given the wide range of grades taught and the specificity of each level of teaching, different activities were proposed for teachers of different levels and one session was given twice by splitting the participants into two groups of different levels.

In the first part of the course, some concepts were provided or recalled and related activities were carried out. The participants performed several tasks intended to widen their knowledge of geometry. Two main topics were addressed: periodic tiling of the plane using Euclidean geometry (wallpaper groups of symmetry) and non-Euclidean geometries. For many participants non-Euclidean geometries were a completely new subject and therefore more time was dedicated to it (two 3-hour sessions about non-Euclidean geometry – one of them split by participant level) and one 3-hour session about wallpaper patterns). After exploring the basic concepts of non-Euclidean geometries, the participants were given tasks to prepare them for the applied project they had to develop during the second part of the course. The participants were invited to draw points, lines (parallel or not), (ordinary or ideal) polygons, to measure angles and to perform circular inversions (reflections) [4, p.314] in the Poincaré disk, with the help of the GeoGebra software. A brief introduction to this dynamic geometry software was also given. Examples of art works from renowned artists using the Poincaré disk were shown and explored. The participants had to solve some tasks at home and make individual presentations during a 3-hour session.

In the second part of the course the participants were asked to develop an individual project that put the theoretical concepts into practice. Each participant was challenged to create their own pieces of art exploring wallpaper patterns and non-Euclidean geometry. The final results were one (or more) patchwork piece with a particular wallpaper symmetry group, one (or more) glazed and coloured ceramic piece by each participant and some styrofoam balls covered with fabrics. This part of the course took two whole days (seven hours/day) of contact hours and the equivalent time of individual work. In the end, a 2-hour session was used to present and photograph all of the pieces produced by the participants.

Results

In this section, photos of a selection of the art work produced are shown and all pictures were taken by one of the authors of this paper with the permission of the participants. The author of each work is identified in each case. All participants were asked to project a ceramic piece using the Poincaré disk or any other non-Euclidean geometry addressed during the course. Most projects were created using GeoGebra.

Figure 1 shows two colourings of the same pattern with 6-fold rotational symmetry around the disk centre. The colouring on the left also has reflection symmetry over six diametral lines. The pattern is based on a tiling of the Poincaré disk with triangles.

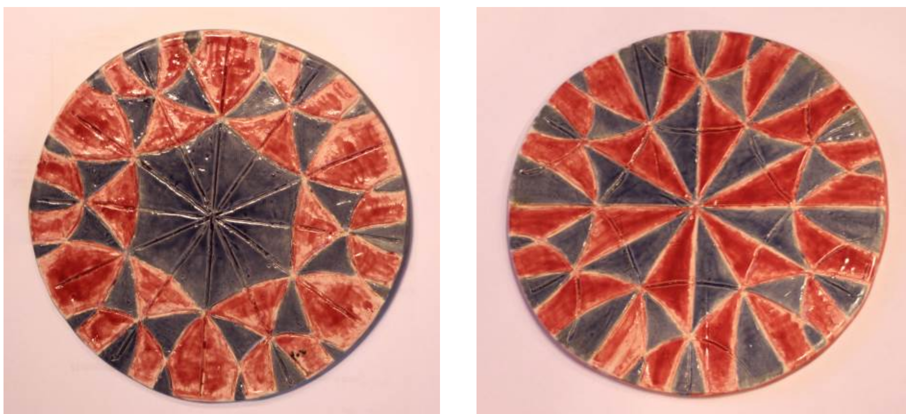

Figure 1: *Ceramic pieces by Ana Berta Póvoa.*

Figure 2 shows two other symmetrical designs which are inspired by similar geometric ideas. Both were obtained from a tiling of the Poincaré disk using, essentially, ideal triangles. The left hand pattern has a central ideal regular triangle whereas the one on the right has a central ideal pentagon. In both patterns ideal triangles emerge outside the central polygon, by bisecting the Euclidean boundary arcs. The procedure is repeated a second time. Both patterns exhibit rotational and reflection symmetry.

Figure 2: *Ceramic pieces by Andreia Hall.*

Figure 3 shows two examples of designs using circular inversions (reflections in the Poincaré disk). On the left, several reflections were done, sequentially, across the sides of ideal triangles, starting from the central one. On the right, only one circular reflection was taken.

Figure 3: *Ceramic pieces by Lúcia Fradinho (left) and Rosa Ferreira (right).*

Figure 4 shows a piece that uses an inside pattern that is inspired by a tessellation of the Poincaré disk with ideal triangles (obtained by reflections across the sides) and on the outside the decoration was inspired by the Poincaré half-plane model.

Note that the Poincarré half-plane model can be obtained from the unit disk model, by "opening" it through the central point and stretching it out to become the upper boundary. At the same time, the disk boundary is converted to a line (as in the Euclidean geometry) which defines the lower boundary. In this model, points lie inside the half-plane. Lines are half-circles whose diameters lie on the boundary line and these correspond to the arcs in the Poincaré disc. Lines are also (Euclidean) half-lines perpendicular to the boundary line and these can be seen as degenerated half-circles with infinite diameter corresponding to the diametral lines of the disk model.

Figure 4: *Ceramic piece by Andreia Hall – Poincaré disk inside and Poincaré half-plane on the outside.*

Figure 5 depicts two designs made of parallel lines on the Poincaré disk. Both designs have reflection symmetry over a diameter.

Figure 5: *Ceramic pieces by Albertina Monteiro (left) and Paula Carvalho (right).*

Figures 6 and 7 illustrate free designs using only lines on the Poincaré disk.

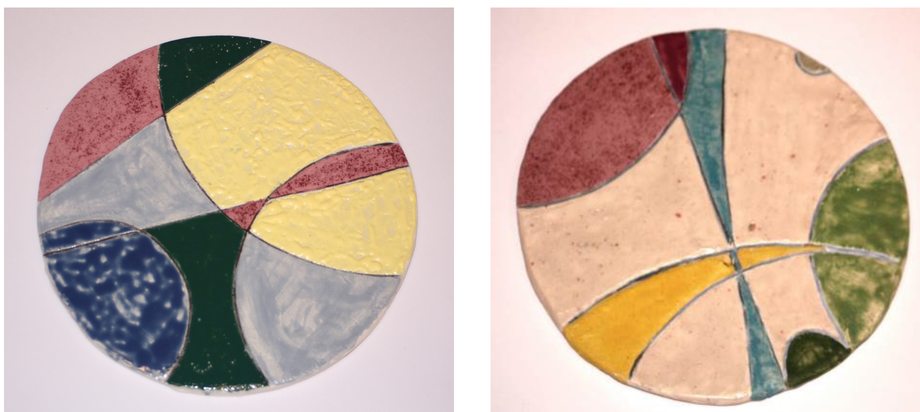

Figure 6: *Ceramic pieces by Rosalina Reis (left) and Manuela Sousa (right).*

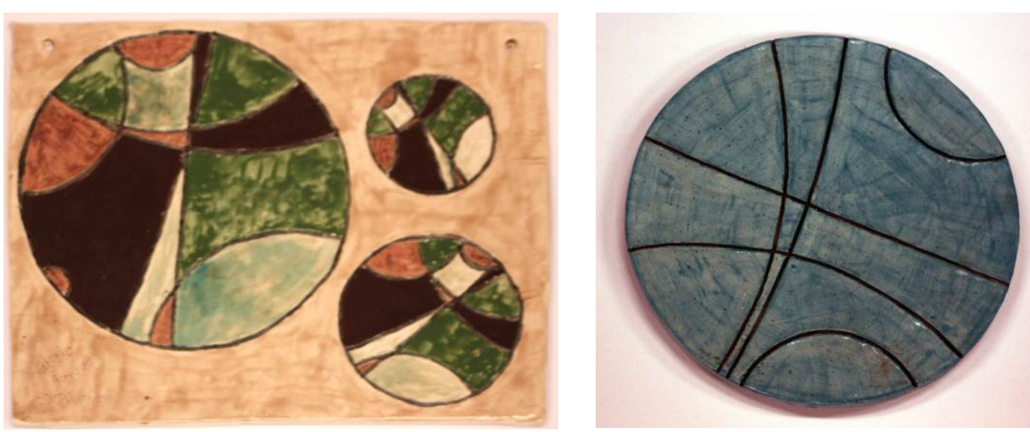

Figure 7: *Ceramic pieces by Anabela Roque (left) and Carla Machado (right).*

Figure 8 illustrates two examples of pieces which use the Poincaré disk more freely. Underlying both designs are Poincaré disks with completely symmetrical divisions. On the left hand plate the lines depicting the umbrellas are lines in the disk and those depicting the handles were freely added. On the plate on the right, the disk boundary was freely deformed but kept the original symmetries of the underlying pattern.

Figure 8: *Ceramic pieces by Cristina Miranda (left) and Ana Paula Moreira (right).*

Some participants applied spherical geometry concepts on styrofoam balls by covering them with fabric using polygons on the sphere to tile them, thus obtaining spherical polyhedra.

Figure 9 shows examples of tilings of the sphere with two-sided polygons. The resulting polyhedra are spherical hosohedra, the first two being regular hosohedra {2,8}.

Figure 9: *Styrofoam balls by Cristina Miranda (left) and Óscar Gomes (right).*

Figure 10 shows two regular tilings of the sphere with triangles, resulting in regular spherical octahedra {3,4}.

Figure 10: *Styrofoam balls by Ana Berta Póvoa.*

Summary and Conclusions

In the professional development course described in this paper, an artistic approach to teaching and learning was adopted. Each participant was given the opportunity to create a piece of art. The participants showed great commitment in the accomplishment of the proposed tasks and projects. Their performance (throughout the entire course) showed that they deepened and/or acquired new knowledge concerning non-Euclidean geometry. They also acquired or improved their competences regarding the use of GeoGebra.

The conclusions withdrawn from this study are in line with those reached by Hall and Pais [5]:

- Overall, the activities developed have proved to be successful examples of interdisciplinary methodologies that bring into the teaching of mathematics usual procedures in the teaching of the arts, associated with the process of artistic creation.
- We believe that these teachers have been strengthened in their capacity to develop multidisciplinary tasks and projects with their students. Although some of the materials and techniques used in the course may be difficult to use in schools, alternative materials may be used instead: patchwork can be done with paper and ceramics may be replaced with paperclay, which is easy to use, dry and colour.
- The teaching methodologies used in this professional development course, involving the interconnection between mathematics and the arts, promote a positive attitude towards mathematics and thus foster motivation to learn it.

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