Symmetries of Intermeshed Crochet Designs

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Abstract

Two forms of symmetry which plane patterns can possess are the traditional wallpaper symmetries and the counterchange symmetries enumerated by H.J. Woods. Intermeshed crochet is a technique which may possess, not only plane symmetries, but symmetries relating the back of the work to the front of the work. We explore how which of these new symmetries are realizable, and in what combinations they can be realized within a single work.

Introduction

Intermeshed crochet is a technique used to create patterns by interlacing two grids of different colors, with one grid brought to the front to exhibit its color with each line of the grid. It is also known as double filet, intermeshing, and interweave, and bears similarities in technique to the more extensively practiced method of double knitting. An abstracted presentation of how this effect is produced is shown in Figure 1. Although work of this sort requires no advanced crochet techniques and consists entirely of crocheting simple grids of double crochets and spaces, this technique remains moderately obscure, described only in a few sources, including a book describing the method and projects implementing it by Tanis Galik [6] and an article by Kyle Calderhead discussing the extension of the technique to hexagonal and triangular grid-pairs [3]; tutorials for intermeshed crochet technique can be found at Galik's site http://interlockingcrochet.com. Calderhead also made use of intermeshed crochet in a contribution to the 2009 Bridges Art Show [2], and cited independent development of the technique and unfamiliarity with Galik's work[4]. The few works published on intermeshed crochet, in both the crafting and mathematical spheres, have been written largely without reference to each other.

One striking aspect of intermeshed crochet, which Galik notes and makes use of, is that there is not only a design of two colors which appears on the front of the work, but a design on the back, which is uniquely determined by the design on the front but which may look very different. Figure 2 illustrates this effect: the front side of this coaster was designed to have the space-filling Moore curve appear in black, while the back of the same work has a patterned collection of black loops which, on casual inspection, do not have any resemblance to the Moore curve. In this regard intermeshed crochet is very different from double knitting, a technique it somewhat resembles. In double knitting, the design on the front may be regarded as a two-color pixelated image, and the design appearing on the back is simply the color reversal of that pixelated image. In intermeshed square-mesh crochet, however, the colored elements are not the interior of grid squares (as a pixelated image would be), but the edges of grids; furthermore, the two grids are offset from each other by half a square. For these reasons, the relationship between the front and the back of the work is more

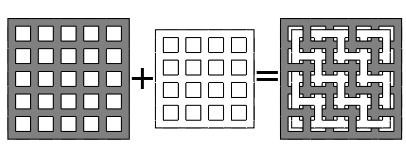


Figure 1: *Conceptual illustration of intermeshed crochet, in which two grids, here depicted in gray and white, are intermeshed to form a zigzag pattern*

(a) *Front of work*

(b) *Back of work*

Figure 2: *An interlocking crochet piece featuring the Moore curve, viewed from both the front and back*

complicated than mere color-reversal. The same two-color pattern formed by bringing different parts of an established grid to the front of the work can also be produced by weaving, using a method described by Ahmed and Deussen as Tuti Patterns [1]. In some work the pattern difference is a liability, as an attractive image on the front of the work can have a considerably less pleasing back side, but in other work the front and back exhibit different images which are both aesthetically pleasing.

Since aesthetic quality can derive from symmetry, it is illuminating to determine how the back of an intermeshed crochet work can appear to be identical to the front, possibly after being subjected to a rotation, reflection, or translation. Considering the context of repeating patterns, which are the usual choices for motifs both in Galik's work and in other crochet pattern libraries, we shall build on the framework and vocabulary of the seventeen crystallographic or wallpaper groups. Notably, this concept has already been broadened into discussing symmetries not only within a single "foreground" color, but also symmetries of color-reversal between a work's two colors (neither of which can be rightly described as foreground or background), in what H.J. Woods [8] identified as the 46 *counterchange symmetries* in the plane. While the crystallographic groups only identify the symmetries which map one design onto an exact duplicate, Woods's work classified patterns according to two distinct symmetries, those which preserved both colors and those which reversed the two colors. Intermeshed grid crochet, since it describes patterns with not only two colors but also two sides, admits four different types of potential symmetries: there are the conventional wallpaper transformations which map a work to an exact duplicate or itself and the counterchange transformations which map a work to a color-reversed duplicate, just as established by Woods's study, but there are also transformations which map a pattern appearing on the front of the work to the location where the same pattern appears on the back, and those which map a pattern appearing on the front of the work to a location where a color-reversed variant appears on the back. This will by no means be a full generalization of the known enumerations of the crystallographic and counterchange groups, since it is both bound to a rectangular grid and implements the peculiar constraints of intermeshed grid crochet. Calderhead's hexagonal grid technique might be applicable specifically to those symmetries which do not invert colors, but a color-inversion symmetry would be unlikely to be realizable with a hexagonal grid, since this method uses different grids for the two colors and a pattern appearing on the triangular grid would likely be impossible to replicate on the hexagonal grid.

Formalization of intermeshed crochet and its patterns

An intermeshed crochet pattern, as mentioned above, is characterized by two intermeshed grids, but all of the information of how the grids are intermeshed can be described by a single grid, by associating each edge of the grid with the status of being on the top or the bottom. For purposes of simplicity, we shall call the grid whose top-or-bottom status we record the "black" grid, which is intermeshed with a "white" grid; as a practical matter, of course, these could be any color. As in the classifications of the crystallographic groups, we will assume every intermeshed configuration is periodic, and can consider our underlying grids as infinite to avoid having to consider the edges of the work. We thus define a set of grid-segments on the integer lattice, which we represent as ordered pairs of ordered pairs:

$$\mathcal{G} = \{((x, y), (x + 1, y)) : x, y \in \mathbb{Z}\} \cup \{((x, y), (x, y + 1)) : x, y \in \mathbb{Z}\}$$

Of equal importance, but more cumbersome to express, is the offset grid which appears in white:

$$\mathcal{G}' = \{((x + \tfrac{1}{2}, y + \tfrac{1}{2}), (x + \tfrac{3}{2}, y + \tfrac{1}{2})) : x, y \in \mathbb{Z}\} \cup \{((x + \tfrac{1}{2}, y + \tfrac{1}{2}), (x + \tfrac{1}{2}, y + \tfrac{3}{2})) : x, y \in \mathbb{Z}\}$$

Since an intermeshed pattern can be associated with a single grid, we may define a pattern by the subset of $\mathcal{G}$ which is on the top of the work.

Definition 1. A *grid pattern* (or *offset grid pattern*) S is a subset of $\mathcal{G}$ (or $\mathcal{G}'$) such that there are linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ such that a segment $s \in S$ if and only if $s + \mathbf{u}$ and $s + \mathbf{v}$ are in S. Following mathematical convention, we will restrict $\mathbf{u}$ and $\mathbf{v}$ to be the shortest vectors in their respective directions which express this periodicity property, and refer to the parallelogram with vertices at $\mathbf{0}$, $\mathbf{u}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{v}$ as the *fundamental domain*.

Every grid pattern describes not only the black design on the front of the work, but also the white design on the front and both designs on the back. These other designs may be derived from S as follows:

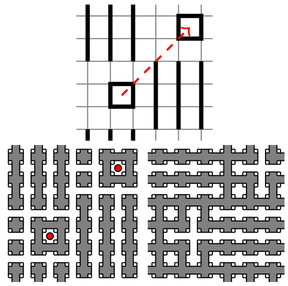
Definition 2. The *conjugate* $\overline{S}$ of a grid pattern S is $\mathcal{G} - S$; likewise the conjugate of an offset grid pattern S is $\mathcal{G}' - S$.

Definition 3. The *dual* S' of a grid pattern or offset grid pattern S is the termwise mapping of the horizontal and vertical segments of S as follows:

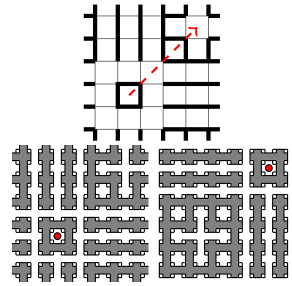
$$((x, y), (x + 1, y)) \mapsto ((x + \tfrac{1}{2}, y - \tfrac{1}{2}), (x + \tfrac{1}{2}, y + \tfrac{1}{2})) \qquad ((x, y), (x, y + 1)) \mapsto ((x - \tfrac{1}{2}, y + \tfrac{1}{2}), (x + \tfrac{1}{2}, y + \tfrac{1}{2})).$$

These two derivations describe which grid lines are visible other than those which are in S and are thus visible in black on the front of the work. Since every segment is either on the front or the back, the black segments appearing on the back are exactly those in $\overline{S}$ (although the visual appearance of the back of the work is actually a horizontal or vertical reflection of $\overline{S}$, since the definition of $\overline{S}$ preserves the orientation of the grid as viewed from the front, which is reflected when the work is flipped over). The dual mapping associates every segment in $\mathcal{G}$ with the unique segment in $\mathcal{G}'$ which crosses it. Thus, any segment in S, in black on the front of the work, has a dual segment in S', which is visible in white on the back of the work. Thus, S, $\overline{S}$, $\overline{S}'$, and S' will be respectively the black segments visible on the front, the black segments on the back, the white segments on the front, and the white segments on the back.

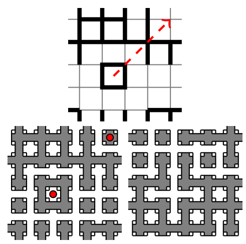
Considering geometric transformations such as translations, reflections, rotations, and glide reflections, we will call a transformation a *standard symmetry* of S if it maps S to itself, a *color-reversal symmetry* if it maps S to $\overline{S}'$, a *complement symmetry* if it maps S to $\overline{S}$, and a *dual symmetry* if it maps S to S'. Standard symmetries thus represent the conventional wallpaper mappings of a pattern to itself and color-reversal symmetries represent the color-swap symmetries in counterchange patters. The two remaining symmetries



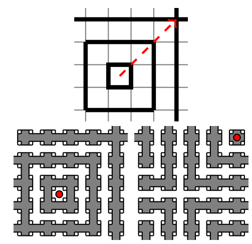
(a) *A standard symmetry under the translation (3,3).*



(b) *A complement symmetry under the translation (3,3).*



(c) *A color-reversal symmetry under the translation (2.5,2.5).*



(d) *A dual symmetry under the translation (2.5,2.5).*

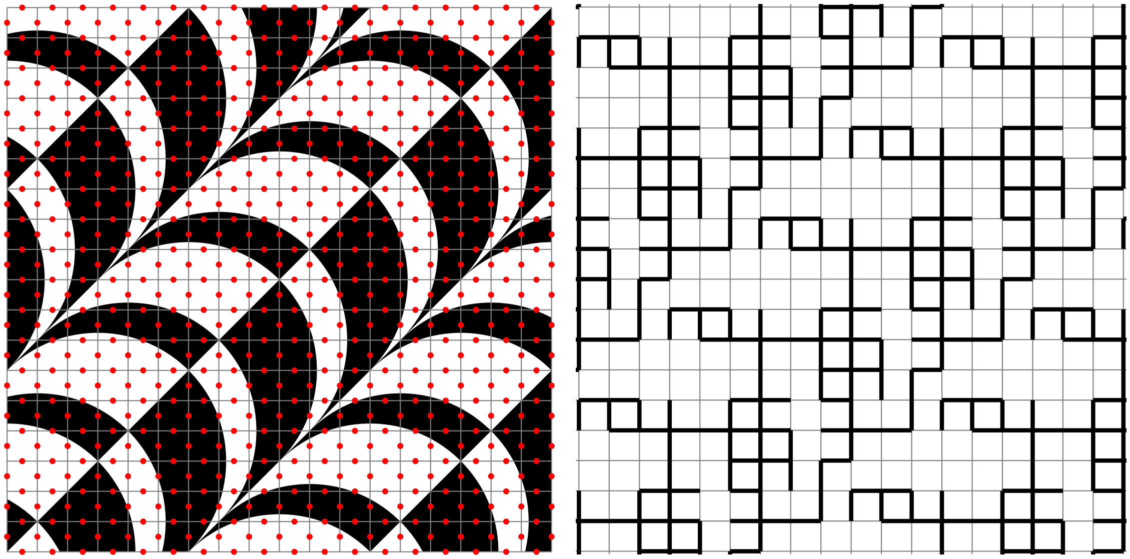
Figure 3: *Exhibits of all four symmetry types as translations, showing the set S, the front of an intermeshed-grid representation, and the back (reflected) of an intermeshed-grid representation. Dots indicate the square at* $(1.5, 1.5)$ *and its images under transformation.*

are present when the design appearing on the back of the work is identical, after perhaps a rotation, reflection, translation, or color-swap, to the design on the front. The complement symmetries represent a transformation from one pattern to an identical pattern on the back, while dual symmetries represent a transformation from one pattern to a color-swapped version of the same pattern on the back. All four symmetries are illustrated in Figure 3, with a dot in the middle of a distinctive square section of the pattern, to illustrate how the four different types of symmetries replicate this pattern in four different ways.

Standard and complement symmetries on the grid

The standard symmetries of a grid pattern must conform to one of the 17 wallpaper groups. Five of these symmetries require threefold rotations which would not map $\mathcal{G}$ onto itself, but the other 12 can all be realized as grid patterns, with only the restriction that axes of reflection must be vertical, horizontal, or 45° diagonal lines so that the reflection maps $\mathcal{G}$ onto itself. Specific examples can be constructed, by overlaying a sufficiently high-resolution grid on a black-and-white image possessing the given symmetry, and then including a segment in S if and only if its midpoint coincides with a black point on the image.

Complement symmetries are the nearest analogue in a grid-design paradigm to Woods's 46 counterchange symmetries, since the two colors in a counterchange symmetry are a partition of $\mathbb{R}^2$ just as S and $\overline{S}$ are a partition of $\mathcal{G}$. As in the case of standard symmetries, the six symmetries which require threefold rotation cannot be realized periodically on a rectangular grid. In addition, we have further constraints on the orientation of reflection axes: as above, reflection axes must be oriented in order to map $\mathcal{G}$ onto itself, but in addition the complement reflections must not map any segment onto itself, which a horizontal or vertical reflection is guaranteed to do, so complement reflections must all be oriented at a 45° angle to the grid lines. Such an orientation is possible for all 40 of the rectangular counterchange symmetries except for two which have counterchange reflection axes at 45° angles to each other: $p4m/p4$ and $p4m/p4g$. Here and henceforth, the Woods symmetries are referred to using type/subtype notation rather than Woods' original typology. The



(a) *Grid superimposed on pattern, with dots indicating segment midpoint.*

(b) *Selection of segments in S by midpoints.*

Figure 4: *The counterchange pattern $pm/p1$ converted into a grid pattern with translational standard symmetry, and translational and reflection complement symmetry.*

type/subtype notation, originally developed by Coxeter for describing two-color frieze patterns [5] and then applied by Washburn and Crowe to plane patterns [7] consists of a type indicated in crystallographic notation of the group of transformations which map each monochromatic region onto another monochromatic region (not necessarily of the same color), followed by a subtype describing the subgroup of transformations which map each monochromatic region onto a region of the same color.

For the remaining 38 counterchange symmetries, the technique of overlaying a grid onto an image and including segments whose midpoint is black in S will suffice to generate examples of complement-symmetry analogues on a grid, as long as the grid resolution is suitable for the purpose. This process is illustrated in Figure 4 on a pattern with the $pm/p1$ symmetry to produce a grid pattern whose only standard symmetries are translational, but which possesses a complement reflection. Note that the image must be oriented so that this reflection symmetry lies along a diagonal of the grid.

Dual and color-reversal symmetries on the grid

Dual and color-reversal symmetries map $\mathcal{G}$ onto $\mathcal{G}'$. The grid $\mathcal{G}$ could be mapped onto itself by any of the following transformations: translating a unit number of steps; rotating 90° in either direction around either a lattice point or a point where both coordinates are half-integers; rotating 180° around a point where both coordinates are individually half-integers or integers; reflecting around a horizontal or vertical axis of the lattice grid or a half-integer step off; reflecting around a diagonal axis on the lattice grid; or using a glide reflection built of these same valid translation lengths and reflections. Mapping $\mathcal{G}$ onto $\mathcal{G}'$, however, requires a different set of rotations and reflections. Translations must be a half-integer number of steps in both directions; 90° rotations must be around points with one half-integer and one integer coordinate, 180° rotations must be around points whose coordinates are both quarter-integers; reflections may only be around axes which are at 45° diagonals offset to the grid by a half-integer step. In addition, the composition of any symmetry (of any type) with itself must be a standard symmetry, the composition of any symmetry of any type with a standard symmetry must be a symmetry of the same type, and the composition of a dual symmetry with a color-reversal symmetry must be a complement symmetry. Furthermore some symmetries are outright

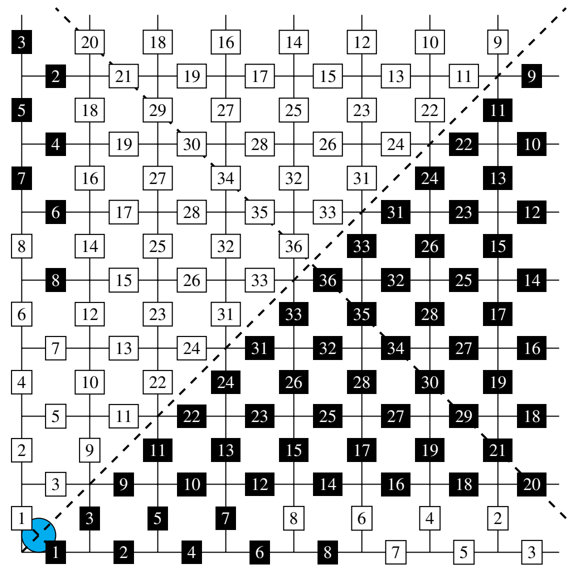


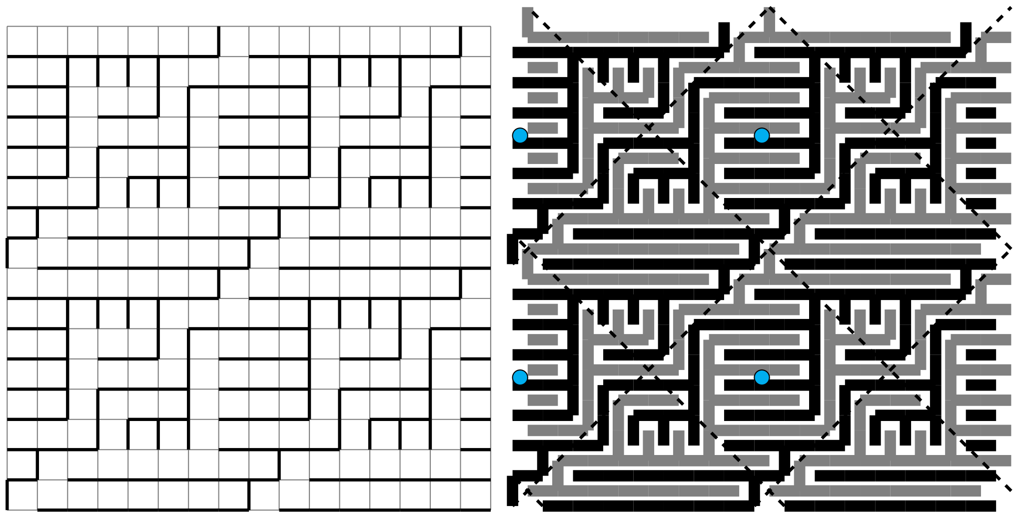
Figure 5: *Application of computer search to subdivide $\mathcal{G}$ into sets conforming to a rotational color-change symmetry around the marked dot, reflectional complement symmetry around the positive-slope dashed line, and reflectional dual symmetry around the negative-slope dashed line. The number i on a white background indicates a segment in S_i, and on a black background in $\overline{S}_i$.*

impossible: there is no pattern with a color-reversal 90° rotational symmetry, because the segment which passes through the center would be required to be both in S and $\overline{S}$.

As a preliminary investigation into this complicated realm, a computational search was conducted for counterchange patterns which possessed at least one dual or color-reversal symmetry; this search was conducted over grid patterns whose fundamental domains are 8×8 squares. Each standard and dual symmetry was assigned and index i and associated with a transformation $f_i : \mathcal{G} \to \mathcal{G}$ such that, in order to obey the ith symmetry of these types, a segment x wold be in S if and only if the segment $f_i(x)$ was also in S. Similarly, each color-change and complement symmetry was assigned an index i and associated with a transformation $g_i : \mathcal{G} \to \mathcal{G}$ such that, in order to obey the ith symmetry of these types, a segment x wold be in S if and only if the segment $g_i(x)$ was *not* in S.

Furnished with these functions, the search decomposed the grid $\mathcal{G}$ into membership-in-S classes as follows. An arbitrary segment was assigned to S_1, and then the functions f_i were applied iteratively to every element of S_1, adding the images to S_1. Then all the transformations g_i were applied to every element of S_1, and the results placed into $\overline{S}_1$. These two procedures were repeated, adding elements to S_1 and $\overline{S}_1$, until no further iterations of the procedure added new elements to S_1 or $\overline{S}_1$, ensuring that both S_1 and $\overline{S}_1$ would be closed under every f_i, and that every g_i maps each element of S_1 to an element of $\overline{S}_1$. Then, among the segments not yet assigned to S_1 or $\overline{S}_1$, an arbitrary segment was placed into S_2, and the same procedure as above used to assign additional segments to S_2 and $\overline{S}_2$. New sets were produced until every segment was assigned to some set S_j or $\overline{S}_j$. If at any point a segment was assigned to *both* S_j and $\overline{S}_j$, the procedure terminated reporting that the set of given symmetries were incompatible.

An example of this procedure, as practiced on an 8×8 grid, is the determination of which sets S possess an 180° rotational color-change symmetry around the point $(\frac{1}{4},\frac{1}{4})$, a complement reflection symmetry across the line $y = x$, and a dual symmetry across the line $y = \frac{1}{2} - x$. These correspond respectively to the functions $g_1(x, y) = (\frac{1}{4} - x, \frac{1}{4} - y)$, $g_2(x, y) = (y, x)$, and $f_1(x, y) = (\frac{1}{2} - y, \frac{1}{2} - x)$, where a segment is described by the coordinates of its midpoint. The algorithm described above partitions the 128 segments in $\mathcal{G}$ into 36 sets S_i and their complements $\overline{S}_i$, as shown in Figure 5. Once this procedure is complete, a set S possessing



(a) *Elements of the set S*

(b) *Symmetries of the two-color pattern: dots are color-change rotations, positive-slope dashed lines are complement reflections, and negative-slope dashed lines are dual reflections*

Figure 6: *A grid pattern possessing symmetries of all four types*

the desired symmetry can be constructed by taking a union of one part from each complement class, i.e. $S = \bigcup A_i$ where each A_i is either S_i or $\overline{S}_i$.

Below, symmetries are denoted using the crystallographic notation of representing a rotational k-fold symmetry with the numeral k and a symmetry under reflection with the letter m. However, modifiers will be placed on these symmetries so as to indicate those which are not standard symmetries: a dual symmetry will be indicated with a prime (e.g. $2'$ for a pattern which possesses dual symmetry under an 180° rotation), a complement symmetry with an overline, and a color-reversal symmetry with both an overline and a prime. As in crystallographic notation, a concatenation of multiple symbols indicates multiple symmetries, e.g. $\overline{2}mm'$ would describe a pattern which has a complement 2-fold rotation symmetry, a standard reflection symmetry, and a dual reflection symmetry (presumably around a different axis).

Patterns which possess at least one complement symmetry, regardless of which of the 38 grid-realizable counterchange symmetry patterns they fall into, can only conform to one of a very small number of dual and color-reversal symmetries: they can possess the dual symmetry under an 180° rotation $2'$, the dual symmetry across an axis of reflection m', the color-change symmetry under an 180° rotation $\overline{2}'$, or the above symmetries in the combinations $2'm'$, $m'm'$, $\overline{2}'$, $\overline{2}'m'$, or $\overline{2}'m'm'$. The computer search result exhibited in Figure 5 indicates that the $\overline{2}'m'$ symmetry (which of necessity possesses reflection complement symmetry, as the composition of $\overline{2}'$ and m') on an 8×8 fundamental domain can give any of 2^{36} patterns, since S is a union of 36 sets, each of which can be either S_i or $\overline{S}_i$. One example of such a pattern is depicted in Figure 6; this particular example was produced by letting $S = \bigcup_{i=1}^{18} S_{2i-1} \cup \bigcup_{i=1}^{18} \overline{S}_{2i}$.

Symmetries in pattern libraries

Much of the text of Galik's *Interlocking Crochet* is devoted to describing patterns to be used in intermeshed crochet. Nine of the patterns are designated as "single designs" where the pattern given appears (possibly subjected to a rotation or reflection) on both sides of the work in the same color; 35 other pairs of patterns are called “double designs”, which are crafted such that distinct designs appear on the front and back. Galik’s classification specifically differentiates between those patterns which possess a complement symmetry and those which do not. The first nine designs can be classified as counterchange patterns, but only exhibit 5 different symmetries: $p1/p1$, $pm/pm(m)$, $p2/p2$, pmg/pmg, and $p4m/pmm$. Since many more symmetries exist, there are likely extraordinary designs which have identical appearance on the back and front which are not yet cataloged.

Among Galik’s double-design pairs, it is unsurprising that almost all of the patterns possess standard symmetries beyond $p1$, mostly $p4m$ and pmm, with a few pg. However, three stand out as possessing symmetry which exploits the grid-duality structure. Design pair 1 & 2 (“Rows/Columns”) has a $p4m/pmm$ counterchange symmetry but also has a translational color-reversal symmetry and a rotational dual symmetry. Pair 9 & 10 (“Lattice Columns/Lattice Rows”) has no complement symmetries but still possesses a pmm standard symmetry and a 90° rotational dual symmetry. Pair 21 & 22 (“Chevron—Light on Dark/Dark on Light”) has pm standard symmetry and a 180° rotational dual symmetry.

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