Portraits from the Family Tree of Plane-filling Curves

Jeffrey Ventrella

jeffrey@ventrella.com

Abstract

The classic dragon curve is an early example of a plane-filling fractal curve with a complex shape. As an exercise in comparing the morphologies of similar curves, a taxonomy is being formulated to enable classification and discovery of new curves. With an eye toward biological metaphor and self-similar aesthetics, a collection of designs is being developed in parallel with the formulation of this taxonomy. It is based on two kinds of complex number integers (forming a square or triangular lattice in the plane) which provide an algebraic framework for categorizing and generating curves. Some techniques for rendering these curves are described, intended to bring out the inherent characteristics of families of curves.

Introduction

Many fractals resemble forms in nature. We find visual similarities and differences in these forms, as we might do when classifying trees, vegetables, or seahorses. This paper presents some new results from developing a classification of plane-filling fractal curves [15]. This exercise has helped to create visual and conceptual tools that are used to discover new curves that may otherwise lay hidden. This includes new discoveries of highly-textured patterns that can tessellate.

The 3-segment generator for the Terdragon is shown at the top-left of Figure 1. Replacing each segment with a copy of the generator results in the shape immediately to its right. Each copy is resized and rotated so that its start and end points coincide with the start and end points of the segment it is replacing. The first 5 iterations of this process is shown. Below that is a splined rendering (using smooth rounded corners) to allow the sweep of the curve to be appreciated. To the right of that is a stylized rendering of a variation of the Terdragon, discovered as a result of developing this classification scheme. The 7-segment generator for this curve is shown at right, with segments numbered to show ordering. The generator crosses itself, forming a loop. The 3rd, 4th, and 7th segments are *rotated by 180 degrees*, as indicated by the half-arrows that are pointing in the opposite direction from the other arrows, relative to the direction of the path. (The result of this kind of rotation is explained later). A splined rendering of the generator is shown at lower-right; it has variable thickness proportional to segment length. This Terdragon variation permits self-crossing without self-overlapping—a feature that can be exploited for visual design: after rendering the curve, closed regions are filled-in with colors to bring out its self-similar structure, while still revealing the continuous line (*sweep*) that traces out its shape.

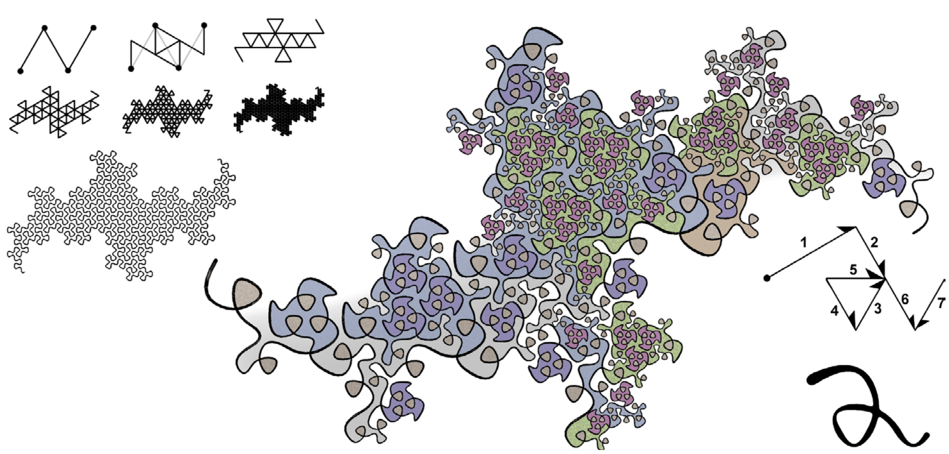
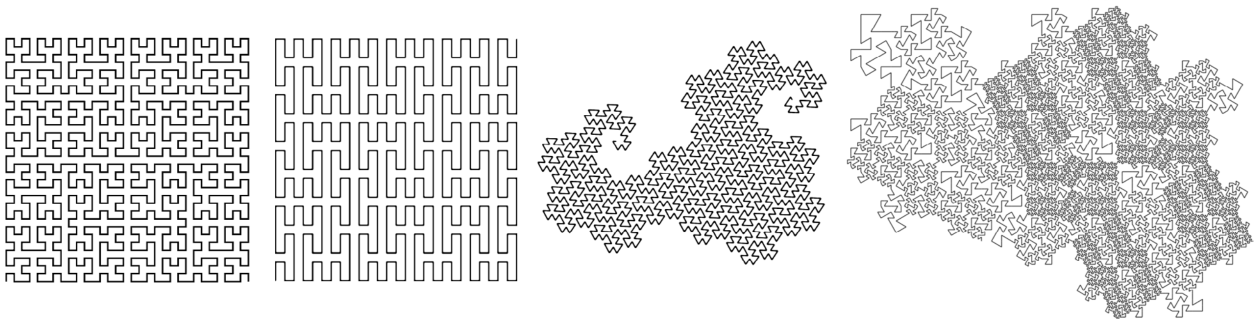


Figure 1: *Left: six iterations of the Terdragon and a splined rendering; right: a stylized variation.*

The subject of space-filling curves [13] is often introduced using the example of the Hilbert curve [7] and the Peano curve [11], shown in Figures 2(a) and (b). These curves fill a square region; they are described as a continuous mapping from the unit interval to the unit square. Space-filling curves can be 2D (plane-filling), 3D (volume-filling), or n-dimensional. Plane-filling curves that fill a square region are practical for many applications, including spatial data representation and scientific computing [2]. Non-square plane-filling curves, while less practical, are often visually striking and bring to mind organic forms in nature, as Mandelbrot suggested when introducing fractal curves to a larger audience in *The Fractal Geometry of Nature* [9]. Figures 2(c) and (d) show two non-square plane-filling curves that are both self-avoiding (a self-avoiding curve never touches, crosses, or overlaps itself). Curve (c) is called “Gosper Swan”. Curve (d) is called “Unravelled Carpet”; it has line segments of varying lengths, which causes the sweep that fills out its shape to have a complex, self-similar structure.



(a) (b) (c) (d)

Figure 2: *(a) Hilbert curve, (b) Peano curve, (c) and (d) two non-square plane-filling curves.*

Plane-filling curves are examples of *fractal curves*, which are curves with geometrical detail at every scale, and a fractal dimension (Hausdorff dimension) between 1 and 2. A straight line has a fractal dimension of 1. A curve with a dimension that is slightly greater than 1 gently meanders through the plane. The higher the fractal dimension, the more curvy the line (at all scales), and the more “space” it fills. Higher-dimensioned curves are more likely to come in contact with themselves due to their more wildly-meandering paths. At dimension 2, a curve can easily become a tangled hairball. In order for a curve with dimension 2 to evenly fill a region of the plane without overlapping with itself or leaving gaps, it must conform to some kind of lattice or grid.

Fractal curves can be constructed using a process called “edge replacement” [12] (also called “Koch construction” or “self-substitution”). Figure 3 illustrates the general concept of edge replacement using an example self-avoiding curve called “Dragon of Eve” (left). To the right is a more typical way to illustrate edge-replacement, using an example curve called “Walking Terdragon”. Both curves shown are based on a generator with 3 segments. The Dragon of Eve conforms to a square grid and the Walking Terdragon conforms to a triangular grid.

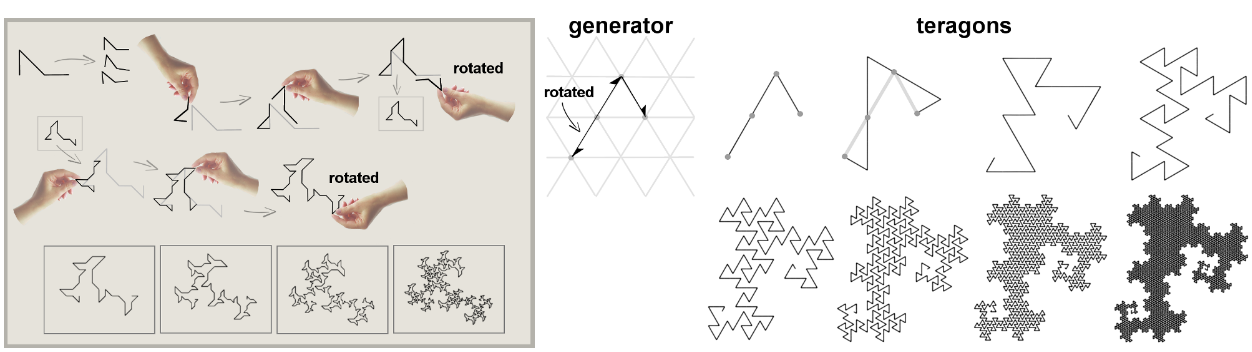


Figure 3: *Edge-replacement in the Dragon of Eve (left) and the Waking Terdragon (right).*

Here is a brief explanation of the process for iterating the Walking Terdragon generator. Replacing each segment with a scaled-down and rotated copy of the generator brings the number of segments to 9. Repeating this process multiplies the number of segments by 3 each time. Starting with the generator, seven stages of edge-replacement are shown—they determine the first eight *teragons*. A "teragon" is a curve or polygon with an infinite number of edges, but the term is used here to denote *orders* of edge-replacement. For instance, "teragon order 8" is the last one in the series shown. It consists of 3^8 segments. In an ideal, Platonic fractal curve, this process is repeated infinitely many times, which is why such a curve can be called "plane-filling". For the purpose of this project, we only need to iterate a handful of times in order to resolve to a sufficient level of visual detail. These might more accurately be called "lattice-filling curves", where the lattice can be of arbitrary resolution.

Notice how one of the segments in each example generator is rotated 180 degrees. This is indicated in the Walking Terdragon generator by the direction of the first half-arrow. This rotation causes the bump in the bottom of teragon 2 to orient downward instead of upward. The rotation is replicated throughout every teragon. In both examples, a rotated segment is critical for these curves to be self-avoiding.

Taxonomy

Since the early classics introduced by Peano, Hilbert, Polya, and Sierpinski, newer plane-filling curves have been discovered by several others, including Gosper [16], Mandelbrot [9], McKenna [10], Dekking [5], Fukuda [6], Ventrella [14], Arndt [1], Bandt [3], and others. As new curves are added to the repertoire, there is opportunity to make comparisons and identify common themes. In the book, "The Family Tree of Fractal Curves" [15], the author describes a taxonomy of plane-filling curves based on two kinds of complex integers. That taxonomy is the basis for these designs. The book can be referenced for a deeper explanation of the taxonomy than what can be presented in the scope of this paper.

Fractal curves are geometrical objects whose complexity grows as a function of iterating some rule or geometrical transformation. They have a few things in common with natural organisms: (1) their "phenotypes" (the visual result of repeated iteration) are more complex and information-rich than their "genotypes" (fractal generators); (2) they lend themselves to classification and comparison, with emergent structures that can be described, compared, contrasted, and appreciated on a metaphorical and aesthetic level. Taxonomies may be possible due to the nature of genesis; in organisms, particular structures emerge as a result of fundamental constraints that come into play during evolution and embryology. In the case of plane-filling fractal curves, square and triangular lattices formed by *Gaussian* and *Eisenstein* integers, respectively, is proposed as a framework of constraints. These lattices have algebraic properties. Like the familiar 1-dimensional (rational) integers, they form a *Euclidean domain*: they are closed under addition and multiplication—the rules of integer math can be applied.

Figure 4 shows the overall morphologies of all plane-filling curves found within the family based on Eisenstein *primes* having norm 3 (spanning the longest distance across the union of two adjacent triangles in the triangular lattice). Their generators are shown at bottom, revealing two unique types of three-unit paths that cover this distance—from start to finish. The generator segments include various transforms: in addition to the 180 degree rotations explained above, they also include *reflections*, and *rotation/reflections*. Thus, there are four unique segment transforms: normal; rotated; reflected; and rotated/reflected. These are illustrated using half-arrows in the diagram at left.

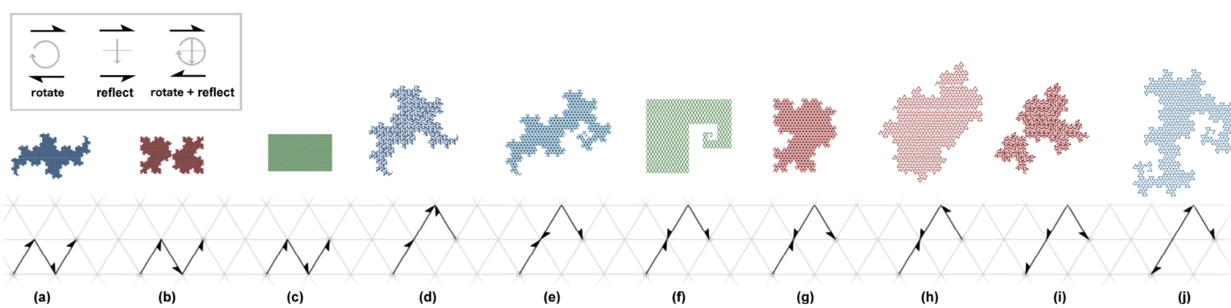


Figure 4: *Plane-filling curves based on Eisenstein integers with norm 3, shown with their generators.*

The curves in Figure 4 are colored according to the classification of their *skin* ("skin" refers to the boundary of the area swept by a plane-filling curve, which can have a fractal dimension of 1 or some higher value < 2). Curves with identical skins have the same fractal dimension and their boundaries appear identical upon arbitrary high magnification. They can fully or partially tessellate. As an example, the Terdragon (a) has three skin-relatives: curve (d), described by Bandt [3], curve (e), described by Karzes [8], and the Walking Terdragon (j).

The classic Harter-Heighway Dragon curve [4], is shown at the bottom of Figure 5. It is classified as a member of a family of curves represented on a power-of-two spiral in the set of Gaussian integers. The spiral is centered at the origin of the complex plane. The "family integer" is an important parameter, as explained in [15]. The family integer of the Dragon curve is the Gaussian integer 1+i, shown as the dot on the right side of the spiral. The Euclidean distance of this integer from the origin is $\sqrt{2}$. The *norm* of this integer (the square of the Euclidean distance) is 2. The spiral represents all powers of this integer. The 2-segment generator of the Dragon curve is shown to the right of the curve. The first segment in the generator starts at the origin (shown with a dotted circle), and the second segment ends on the family integer. Note that its second segment is rotated.

Multiplying the family integer by itself results in the integer shown in the dot at the top-right of the spiral. This integer is 2i and it has norm 4. It has more degrees of symmetry and can thus express more variation on a theme, including a variation of the dragon curve, shown above and to the right of the spiral, along with its generator. Multiplying 2i by the family integer results in the integer -2+2i, with norm 8, shown as the dot at the left of the spiral. An associated variation of the dragon and its generator are shown at top-left, and a stylized version is shown at the right of the figure. More power-of-two dragon relatives lie along this spiral, awaiting our admiration and the opportunity for stylized portraiture.

The teragons in Figure 5 are splined to visually separate the contact points. The dragon curve is a fully lattice-filling curve: visiting every lattice point in its body twice during its sweep, except at its boundary. The higher dragons shown are *partially* self-contacting. The splined rendering helps to reveal the spatial polyrhythm of the sweep.

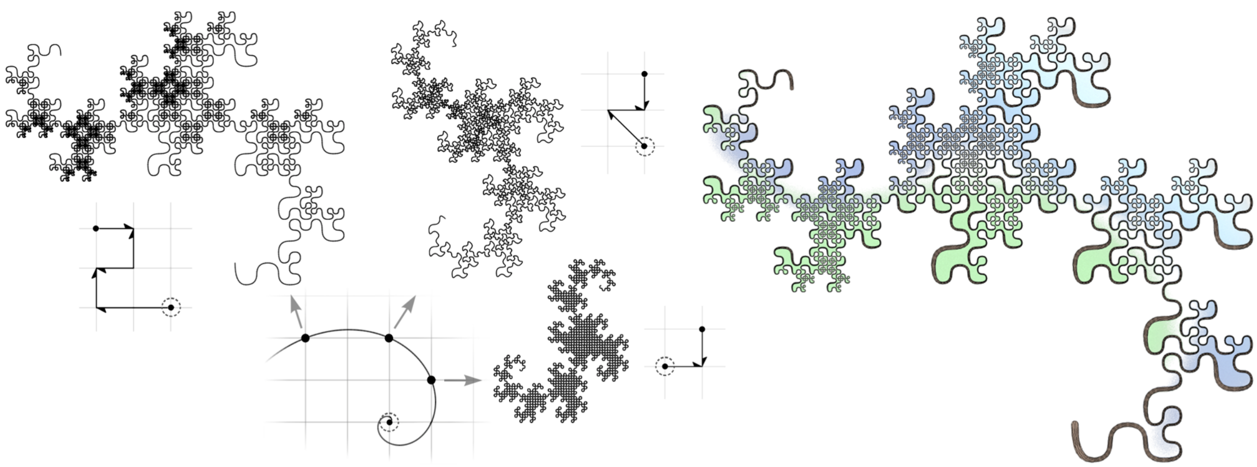


Figure 5: *Higher-power variations of the classic Dragon curve are members of power-of-two families.*

Eisenstein Families

The Eisenstein integers form a triangular lattice. Analogous to the Dragon curve, which can be represented in the set of Gaussian integers, the Terdragon is a member of the first family lying on a *power-of-three* spiral within the set of Eisenstein integers. The members of that family are shown in Figure 4. For present purposes, all families can be referenced by their Euclidean domain (Gaussian vs. Eisenstein) and their norm. For instance, the "E13 family" refers to the family in the set of Eisenstein integers with norm 13.

As with rational integers, Gaussian and Eisenstein integers have prime numbers (numbers that cannot be expressed as the product of two integers that both have a smaller norm). Examples of small Eisenstein primes include integers with norms 3, 4, and 7. Prime families have no curves with variable-length segments; all of their segments have length 1. The segments of a generator for a plane-filling curve are represented as an array of complex numbers whose sum is equal to the family integer. The integers in the generator of a curve of a prime family can only have norm 1 (length 1), and the number of integers in that generator is equal to the norm. A few example curves from the E4, E7 and E9 families are shown in Figure 6. Their integers are indicated as dots—with norms—at upper-left. Included in the E4 family is the Sierpinski Arrowhead curve (g) (with dimension < 2). The E7 family includes the Gosper curve (e) and the Gosper Swan (f) which is a self-avoiding curve with Gosper skin and a torso having the same general profile as the Gosper curve. Several other curves of the E7 family also have Gosper skin.

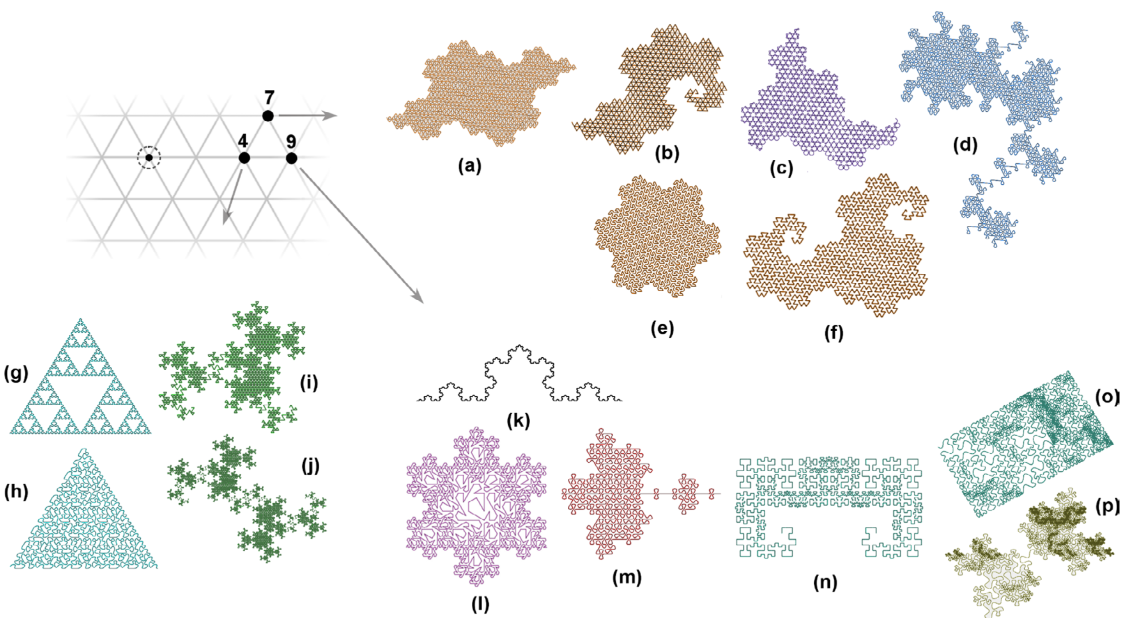


Figure 6: *Three Eisenstein families with a few representative curves.*

The E9 family is not a prime family (since it has divisors with norm 3). Furthermore, it is an example of a "power family"—the first power of the E3 family integer. It includes various higher-order relatives of the Terdragon, such as the stylized curve shown in Figure 1. It also includes the Koch curve (k) (not a plane-filling curve, but a fractal celebrity worth mentioning). It also includes Mandelbrot's Snowflake sweep, which fills the Koch snowflake (l), and many other curves, a few of which are shown at lower-right.

Figure 7 shows four curves of the E9 family whose teragons all resolve to the same shape—which is oddly asymmetrical. Their generators include segments with various transforms. Notice the interior textures, which result from their different sweeps.

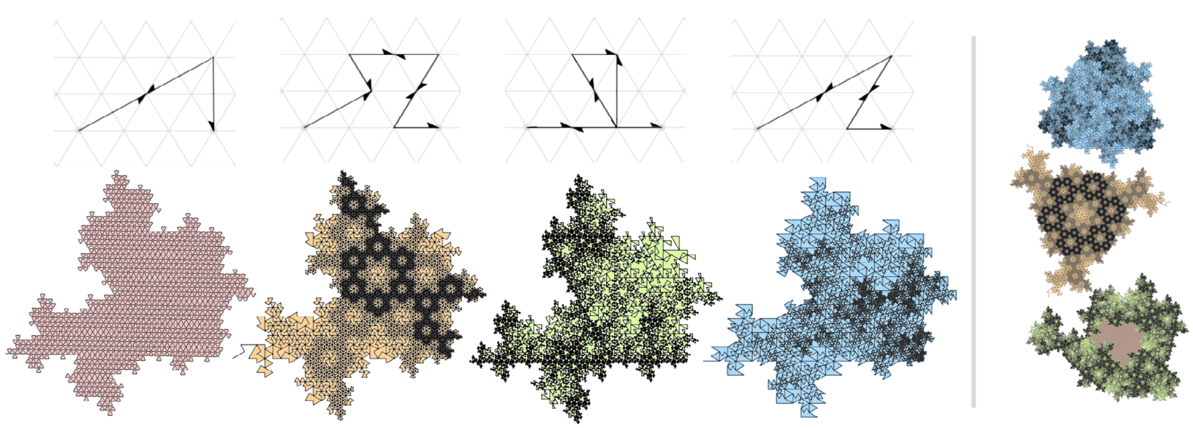
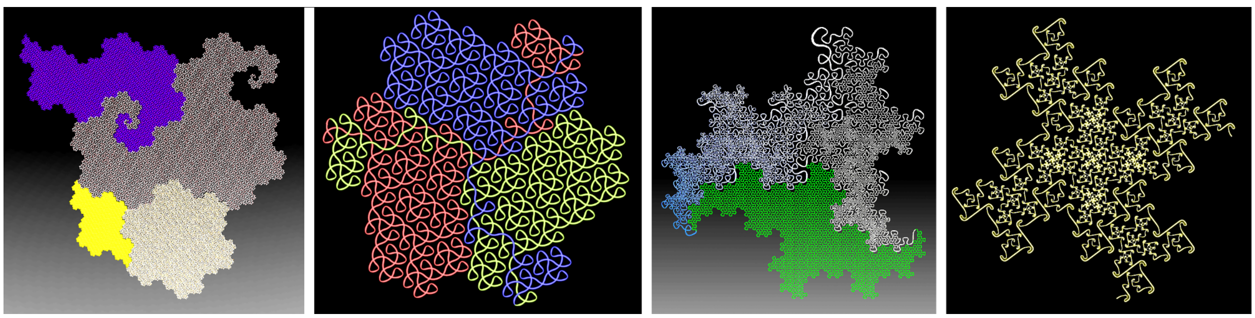


Figure 7: *Four curves from the E9 family demonstrating different sweeps through the same shape.*

The ancestor of these curves is shown in Figure 4(g). The left-most curve in Figure 7 is a 3x scaled version of the ancestor. The integers of its generator all have the same norm (3), which accounts for its smooth, even texture. A few experimental tilings are shown at right. These are just a few ways to express the complex nature of these curves.

Portraits

There are endless ways to express the relationships between individual curves and families of curves. Some examples are given in Figure 8. Image (a) shows a family portrait of curves from the E7 family that can tessellate because they have Gosper skin. Image (b) shows three identical self-crossing E7 curves engaged in an intimate embrace that fills the interior of the Gosper Island. Image (c) shows the Terdragon (green) tessellating with two related E9 curves to form a single continuous curve. Image (d) shows a member of the G25 family, which is a variation of Mandelbrot's Quartet of the G5 family.



(a) (b) (c) (d)

Figure 8: *(a-b) tessellations of curves from the E7 family; (c) tessellations of curves of the E3 and E9 families; (d) a variation of Mandelbrot's Quartet from the G25 family.*

Curves with Dimension < 2

The design shown in Figure 9 features two close relatives of the E9 family…touching fingers. It is rendered as a single continuous curve that starts at the left and ends at the right. These are higher relatives of the Koch curve, which may account for the similarity to the top profile of the Koch curve. The first four teragons of each curve are shown above each curve, and above that are the first four teragons of the Koch curve—for reference. Notice that both of these curves are based on the same generator shape. The reason they are subtly different is because they have different kinds of transforms applied to their segments.

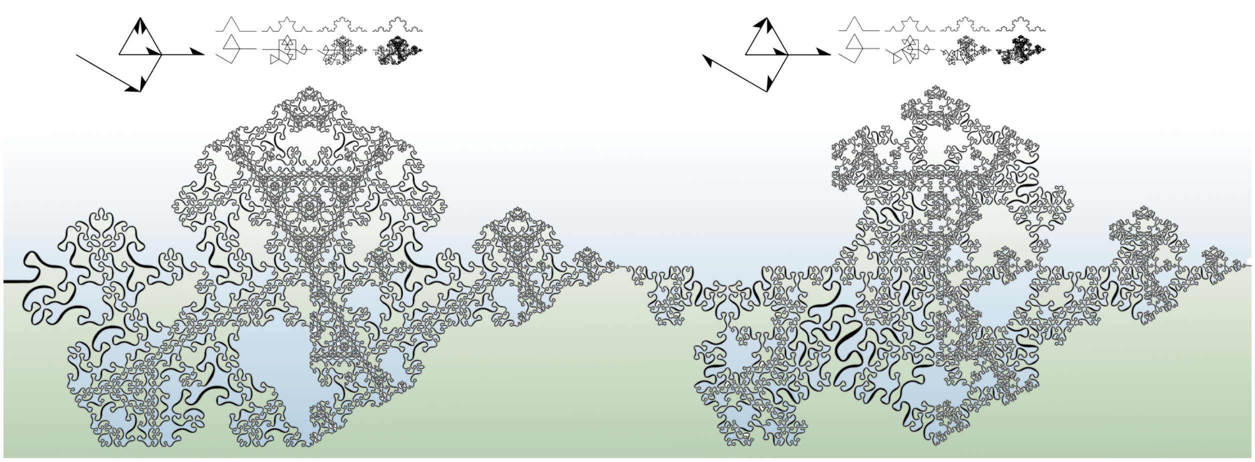


Figure 9: *Two relatives of the Koch curve from the E9 family, touching fingers.*

While these are not plane-filling curves (having dimension ~1.892) they expresses similar kinds of underlying structure as related plane-filling curves. Curves with dimension slightly less than 2 are often more visually complex, having more degrees of freedom in their sweeps, and leaving gaps that express their self-similar structure. These two curves are splined to separate the contact points. Seeing these two closely-related curves together in the same image invites the eye to wander around and appreciate their common traits and also their differences.

Techniques

Figure 10 shows six designs using various coloring and compositional techniques. Image (a) shows a splined version of curve 3 from Figure 7. The lower region of the area defined by the curve (as a boundary) is filled-in to create a strong figure/ground distinction. Image (b) is a tesselation of three identical curves from the E9 family, which are variations of the Walking Terdragon. This tesselation causes the curve to close so it can be color-filled. Image (c) is a member of the E16 family called "Tsunami curve" which is filled in a similar way as image (a). Image (d) shows a member of the G8 family called "Brainfiller"; a gradient tool has been applied over masked regions on either side of this curve. There is no distinction between inside and outside, so the color gradient creates ambiguity. In most cases, the cleverness of the curve's sweep is clearly expressed (inspired by a degree of reverence). Image (e) shows a member of the E25 family that has a color gradient applied in a similar way to (a). Image (f) is a curve from the G8 family that is related to the Dragon of Eve shown in figure 3. All designs use splines except for (b) which is already self-avoiding.

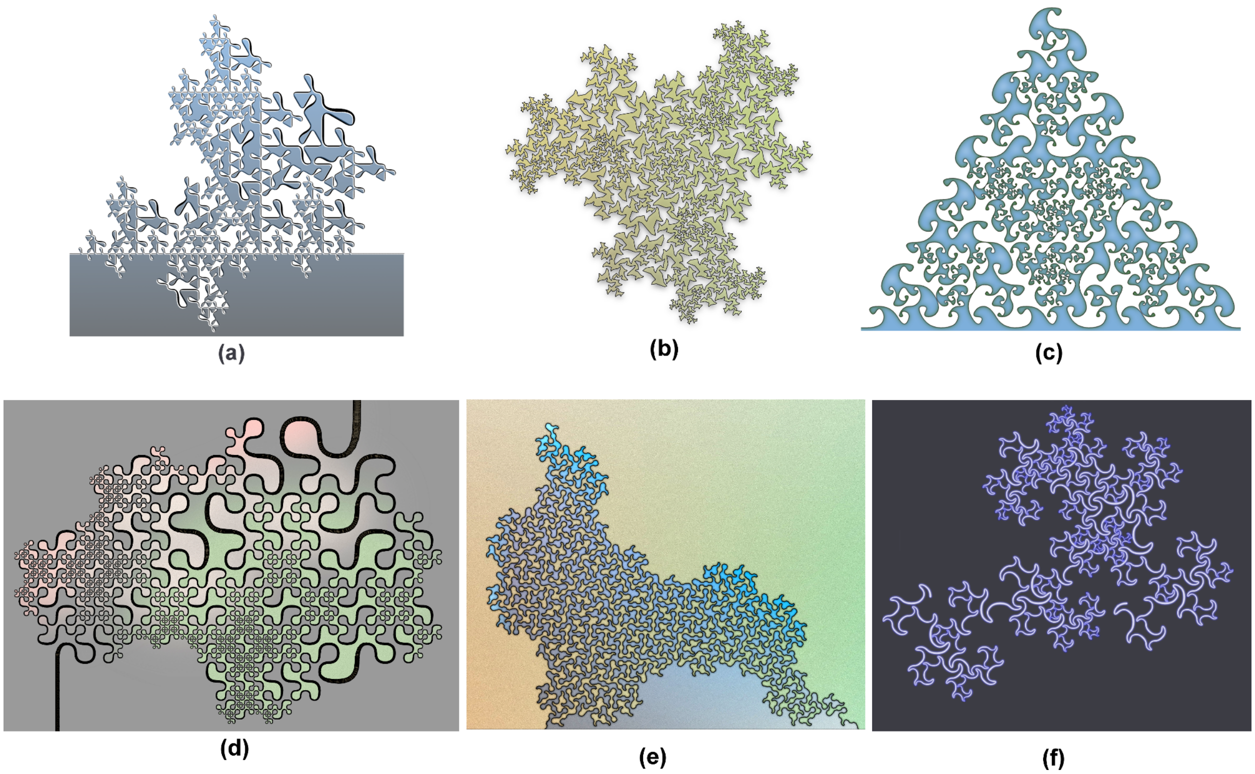


Figure 10: *Various coloring and compositional techniques used in various curves.*

Summary and Conclusions

As a study in aesthetic morphology, this math/art project explores a set of visually complex objects (edge-replacement plane-filling curves) exhibiting features that can be compared and contrasted. They can be appreciated both visually and algebraically. Complex integer lattices are proposed as a framework for classifying these curves, and also for discovering/generating new ones. This exploration has resulted in a large and growing collection of visual designs (stylized portraits of unique curve individuals and family embraces). Most of the classic plane-filling curves from the literature can be classified within this taxonomy, and thus included in family portraits. This is a work in progress, motivated by mathematical curiosity and an effort to build a collection of visual designs that express self-symmetry, biological metaphor, and morphological hierarchy. This project may never have an obvious conclusion, since infinitely many unseen plane-filling curves await our aesthetic gaze and mathematical analysis. Once these curves are found, there will inevitably be more ways to express their unique characteristics—using visual aesthetics to express their mathematical beauty.

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