Exploring the Geometry of Music with Technology: Rhythm, Scales and Temperament via Geogebra

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Abstract

This workshop introduces the use of applications written in the program Geogebra that allow the users to investigate the mathematical and sonic landscapes given via musical notions of rhythm, scales, and temperament. Individual and group activities in the workshop are designed to inspire participants to make their own independent investigations of these landscapes. With this workshop we also hope to bring to the mathematical, musical, and educational communities new options to engage students and practitioners in mathematical and musical explorations.

Introduction

Our goals in this workshop are several. First, to introduce to musicians, mathematicians and educators to the usefulness of the modern technological program Geogebra and short programs written therein ("apps") to the analysis of a selection of the various roles that mathematical ideas play in musical theory. Additionally, we wish to demonstrate with specific examples and activities how these technological tools may be employed to further the development of certain mathematical ideas relating to music. We also wish to introduce educators to these tools in order that that they may train students to make their own independent explorations of the many applications of mathematics to music and also facilitate student's explorations of various sonic landscapes. Our principal context will be the study and development of rhythms and scales, traditionally at first and then moving on to an algorithmic method inspired by the Euclidean algorithm developed by E. Bjorklund in [1] and whose applications to rhythm is discussed by him at length in [2]. As remarked in [2], without presented detail, much of the work presented therein can also be also applied to scales, and for this workshop, we have developed further Geogebra apps that carry out these previously undeveloped applications, in addition to providing the users with a new tool to carry out their own independent explorations of these sonic landscapes. These applications are available in [4]. We note many such algorithmic approaches to rhythms and scales exist in the literature and some have been presented at previous Bridges conferences, too many to list in this short article. Fortunately, an extensive bibliography for the subject can be found in [7]. We note that all images in this paper are screen shots of our applets.

Why Geogebra?

We chose to use the program Geogebra because it is cross platform, open source, user friendly, and based on a simple and easy to understand command language. We particularly like the fact that users can understand precisely how each object in an app is defined with a minimum of language learning. Storing the actual apps on the cloud means further that users can be encouraged to take them apart, change definitions, and so forth. If an app is broken in the process, it can easily be reloaded. Finally Geogebra includes a limited interction with the JFugue musical programming language. This provides an excellent introduction to algorithmic music and allows for MIDI sounds to be used when appropriate, but we have attempted to avoid "nice " sounding instruments in favor of raw sine waves for some of the apps, because the mathematical aspects are easier to hear in these cases. On the other hand, our interest is ultimately in exploring some of the aesthetic issues wrapped up in scale choices. As such some of the aplets make use of JFugue and MIDI instruments.

Workshop Program and Content

The workshop begins with a short review of the traditional mathematical constructions of the Pythagorean, Just and Equal Tempered scales and a brief discussion of the importance of temperament as developed in [3]. The Geogebra apps developed for that discussion are re-introduced and utilized in various group and self-directed activities designed to give participants and intuitive feel for the fundamental musical issues surrounding scale construction and employment. Fundamental here are the notions of consonance and dissonance, in lay language, which families of notes that form harmonies that are pleasant sounding and families of notes that are best described as “alarms” that grab one’s attention with jarring sounds. Related and also explored in this first part of the workshop is the notion of *modulation*, the ability to move between musical keys (this is the musical “root”) within a single musical piece. The apps demonstrate in user decidable ways just how the mathematics underlying the various traditional methods of scale construction (i.e. the Pythagorean, Just and Equal Tempered constructions) either inhibit or facilitate this issue, again under various user determined assumptions. An important take-away from this discussion will be the notion of *even distribution*.

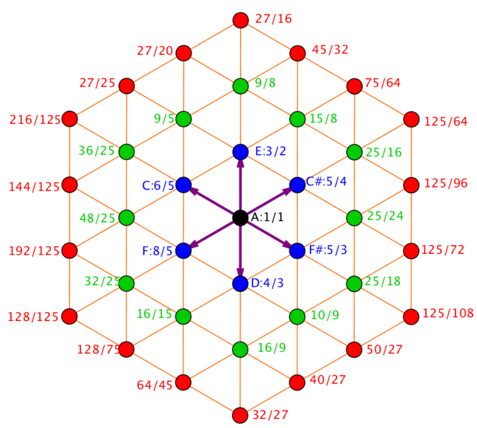

Figure 1: *An applet exhibiting a hexagonal lattice of just note intervals based on the 5-limit tuning. Each note can be played by touching the corresponding point.*

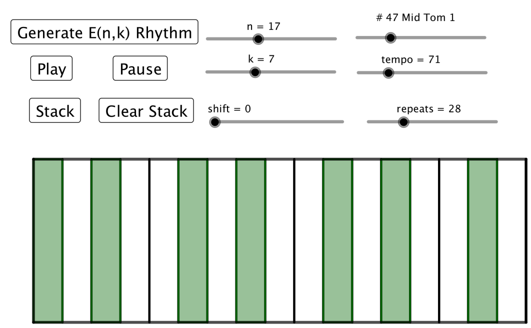

Figure 2: *An applet to generate and play Euclidean Rhythms with MIDI percussion instruments.*

In the second part of the workshop we combine the notion of even distribution with the principle of *algorithm generation.* The literature provides many distinct choices for particular algorithms that generate musical objects. In an effort to introduce these concepts in an intuitive way, at this point we change our context from the development of *scales* to that of the development of *rhythms*. Incredibly to many, an idea as old as ancient Greece has an application in modern music theory. We present an algorithm that closely mirrors the Euclidean algorithm that will allow us to generate rhythms and subsequently, scales. The algorithm itself was discovered by the mathematician E. Bjorklund [1], as part of the investigation of pulse optimization in neutron accelerators; the algorithm works essentially by “spreading things out as evenly as possible.” How it works and its relationship to the Euclidean algorithm are perhaps best understood through an example. Here is one where each step of Bjorklund’s algorithm is paired with its corresponding step in the Euclidean algorithm.

Euclidean Algorithm	Euclidean Rhythm
13 = (8 x 1) + 5	[0] [0] [0] [0] [0] [0] [0] [0] [1] [1] [1] [1] [1]
8 = (5 x 1) + 3	[01] [01] [01] [01] [01] [0] [0] [0]
5 = (3 x 1) + 2	[010][010][010] [01][01]
3 = (2 x 1) + 1	[01001][01001] [010]
	[1001010010100]

The resulting string of 0's and 1's now has the property that the 1's cannot be more evenly distributed.

It turns out that there is an abundant supply of well-known world rhythms that can be generated by Bjorklunds algorithm, as we recall from Toissaint [6]. The idea is to first let *n* be the total number of time units in a repeating interval (for instance, the number of beats in a measure) and let *k* be the number of onsets in that interval. Then the distribution of onsets according to Bjorklund's algorithm is called a *Euclidean rhythm* and is denoted *E(k,n)*.

In Figure 3, we see an illustration of the situation for 8 onsets distributed through 21 pulses. The diagonal line intersects the horizontal line corresponding to each onset at the point of exact even distribution. In this case, staring with the first pulse (pulse 0 along the x-axis), the onsets would occur at regular intervals of 21/8 pulses. Of course, if we are restricted to the 21 pulses corresponding to the x-coordinates 0 through 20, then this is not possible. The closest that we could get to this ideal would be to choose the pulse closest to this point of intersection for each onset. In the figure, onsets corresponding to the Euclidean rhythm E(21,8) appear as diamonds and exhibit this very property, assuming that we start our counting on the rightmost onset. We will see how this applies to music in the next paragraph.

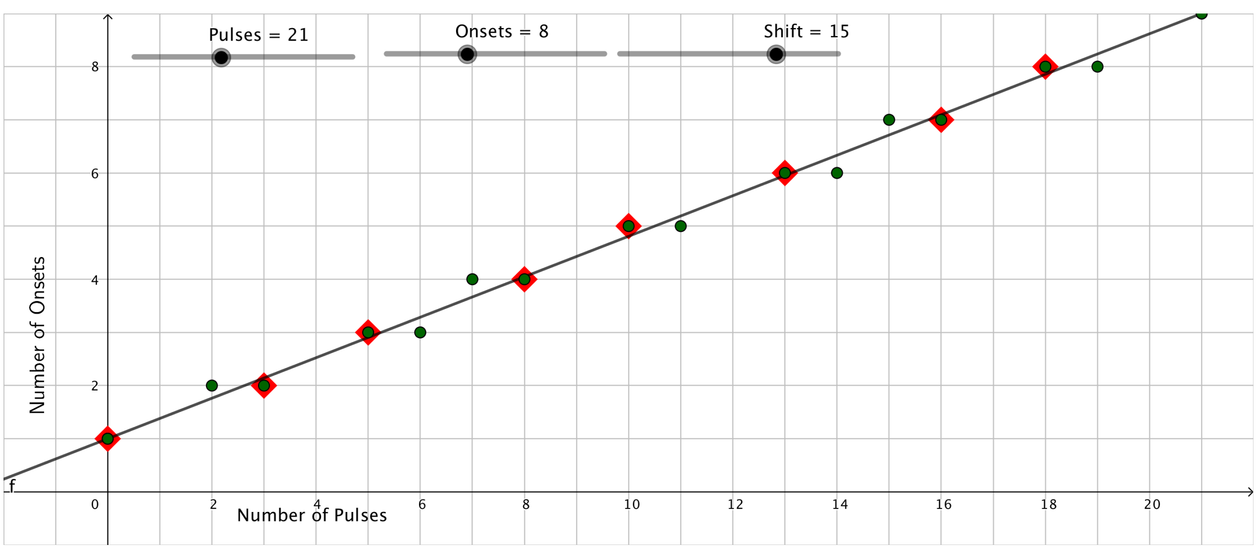

Figure 3: *The distribution of the Euclidean rhythm E(21,8)*

One of the most famous rhythms in the world, called the *Habanera* in the USA and *tresillo* in Cuba, is a Euclidean rhythm, specifically *E(3,8)* that we can view pictorially as: [x . . x . . x .] where 1's are now x's and 0's are .'s. If we disallow periodic rhythms, i.e. considering only cases where *k* and *n* are relatively prime, and identify all rhythms related to each other by a cyclic permutation, a surprising number of famous rhythms are described. Here are a few, as detailed in Toissaint [6]: the rhythm *E(3,7) =*

[x . x . x . .] which is a *Ruchenitza* rhythm in Bulgarian folk music and also the rhythm found in the Pink Floyd song "Money." The rhythm *E(4,11)* = [x . . x . . x . . x .] is employed by Frank Zappa in his work *Outside Now*. The rhythm *E(5,6)* = [x . x x x x] is a popular Arab rhythm known as the *York-Samai* pattern. The rhythm *E(7,12)* = [x . x x . x . x x . x .] is a rhythm of the *Ashanti* people of Ghana. The rhythm *E(5,16)* = [x . . x . . x . . x . . x . . .] is essentially the *Bossa-Nova* rhythm of Brazil occurring as a cyclic permutation of the popularly familiar *Bossa-Nova* rhythm, denoted by [x . . x . . x . . . x . . x . .]. Next is the rhythm *E(9,16)* = [x . x x . x . x . x x . x . x .] This rhythm has various forms starting at different onsets. If started on the fourth onset, it is a West and Central African rhythm. When started on the second to last onset, it is the *Ngbaka-Maibo* rhythm of the Central African Republic.

As can thus be observed, Euclidean rhythms are ubiquitous worldwide and are employed in many different styles of music. Why is this so? One possible reason is that the onsets are evenly distributed. Another is the fact that Euclidean rhythms have some other unique properties, properties as discussed by Demaine, et. al in [5]; in particular, the distances 1,2, ..., floor(n/2) appear a unique number of times when looking at the distances between onsets (where going off the right edge means reappearing on the left edge). For example, the rhythm [x x x . x .] has distance one appearing twice, distance two appearing three times and distance three appearing once. This is a technical item, but to most western listeners corresponds to aesthetically pleasing sound.

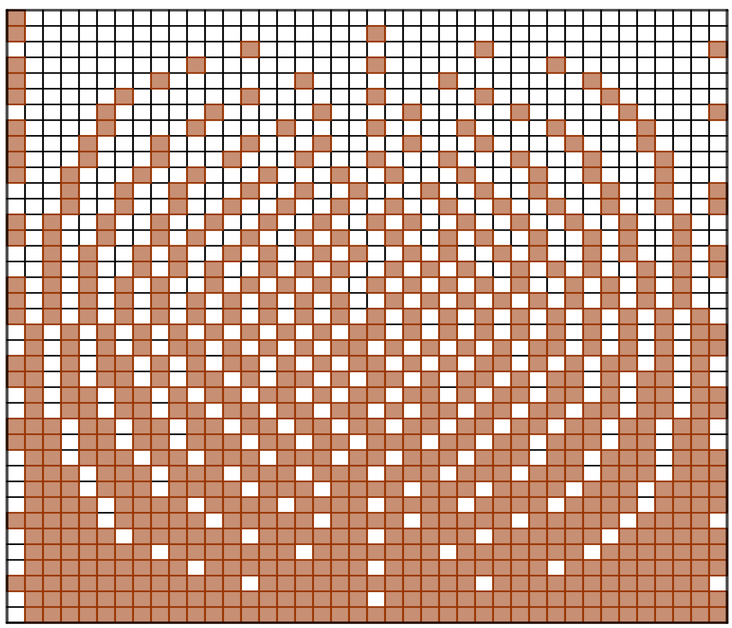

Figure 4: *A visualization of all Euclidean Rhythms with 40 beats.*

Moving to the final part of the workshop, as suggested but not developed by Bjorklund in [2], we explore how Bjorklund's algorithm applies to scales in virtually the same way as rhythms. One begins with a division of the octave into uniformly distributed pitches (like the half-steps in a 12-note equally tempered scale). This division is analogous to the interval of time in a rhythm discussed above. Then there is a sequence of x's or .'s determining whether that pitch appears in the scale or not. For example, we could have: [x . x . x x . x . x . x]. This is the familiar 12-tone major scale. It is clear we could easily define Euclidean scales completely analogous to Euclidean rhythms. Interestingly, the 12-tone major scale is a Euclidean scale corresponding to the Euclidean rhythm *E(7,12)*. From here, it should be clear that all of the traditional modes built by shifting the major scale are also examples of E(7,12) scales. Of course, the same is, somewhat trivially, true for the whole tone scale and the primary notes of the diminished and augmented scales as these scales evenly distribute 6, 4, and 3 notes throughout the 12 semitones respectively. More significantly, the complete diminished scale is the E(8,12) scale. Some other traditional scales can be similarly derived via Bjorklund's algorithm by filling in a smaller sequence with a repeating tonal pattern. Thus, for example, the diminished scale is also the E(4,8) sequence with each non-empty position filled in with a pair of notes one half step apart from each other. Less regular symmetric scales like [x . . x x . . x x . . x] are still "evenly" distributed. But now, it is not a single note that we distribute, but a pair of notes. E.g., if we applied "xx" to E(4,8) = [x . x . x . x .] we get this pattern. In [7], Toussaint focuses on characteristics of "timeline" rhythms that appear as a repeating pattern underlying many songs. He notes that often rhythms that deviate slightly from maximally even ones are particularly appealing in this role. This follows the common musical themes of tension and release that also apply to harmony and melody. Toussaint coins the phrase "almost maximal evenness" to describe rhythms that are close approximations of Euclidean rhythms. These rhythms can be obtained from their maximal counterparts by shifting beats as follows. First, compute the continuous even distribution given by i*n/k for i = 0,2,...,k-1. Then the i-th beat of the Euclidean rhythm will always be either the ceiling or floor of the i-th element of this sequence. A valid shift flips a ceiling to a floor, or vice versa. In Toussaint's analysis, such shifts add excitement to the rhythm by creating tension and release.

As is the case with rhythms, scales come in a bewildering variety. Even within western classical music there are many common scales that are not Euclidean. These include the (ascending) melodic minor and harmonic minor scales, with intervals [x . x x . x . x . x . x] and [x . x x . x . x x . . x] respectively as well as "symmetric" scales based on repeating interval patterns such as [x x . . x x . . x x . .]. Outside of western music one finds scales far less evenly distributed, such as the scale used in the Todi raga in classical Indian music with intervals [x x . x . . x x x . . x].[1] The associated Raga is considered by many to be haunting and beautiful. [2] This scale is only two shifts away from E(7,8).

Summary and Conclusions

These ideas open up many areas and avenues for exploration. As examples, we have designed a few Geogebra applets that mimic a simple keyboard. The user is able to set values for n and k and play the E(k,n) scale on the keyboard. In a second version, the user can supply a list of indices and play the scale in n-ET defined by those indices. Note that the coloring of white and black keys on the piano exhibits the E(7,12). In a third version, we generalize this fact to a n-ET keyboard with white and black keys distinguished by E(k,n). These applets can then be run on a touchscreen device like a tablet or phone creating a simple virtual keyboard.

Geogebra provides a primitive, but effective means of exploring these and other sonic landscapes. The point of our workshop is to show how these mathematical and musical ideas can be simultaneously explored without extensive background in coding or music theory. We note that one can also use music programming languages like JFugue or cSound to explore these ideas, however these languages are complex. While this allows the programs to be very expressive, it also means that they may not be suitable for those without programming experience or the inclination to learn such. On the other hand, all of our apps are based on mathematical, musical, and algorithmic ideas that should be accessible to most

[1] Accessible accounts of Indian music theory are not easy to find. The following website contains some nice articles: chandrakantha.com

[2] We especially recommend Ali Akbar Kahn's famous performance found at https://www.youtube.com/watch?v=nKnYpnszjFk

middle/high school students and college undergraduates without any specific background knowledge. One hope is that such experiments might inspire others to explore these more expressive, but less accessible, alternatives.

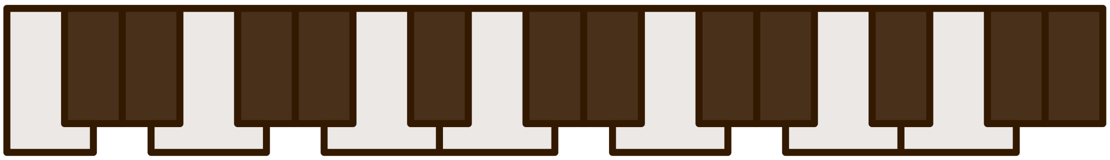

Figure 5: *A playable keyboard layout for 19ET based on E(7,19).*

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