Sevenfold and Ninefold Möbius Kaleidocycles

Michael Grunwald, Johannes Schönke, and Eliot Fried

Okinawa Institute of Science and Technology, Okinawa, Japan
michael.grunwald@oist.jp, johannes.schoenke@oist.jp, eliot.fried@oist.jp

Abstract

In this workshop we will build paper models for a completely new class of kaleidocycles. Kaleidocycles are internally mobile ring mechanisms consisting of tetrahedra which are linked through revolute hinges. Instead of the classical, very popular kaleidocycles with even numbers of elements (usually six, sometimes eight), we will build kaleidocycles with *seven* or *nine* elements and several different surface designs. Moreover, we will learn that it is possible to construct kaleidocycles with any number of elements greater than or equal to six. These new kaleidocycles share two important properties: They have only a single degree of freedom and the topology of a threefold Möbius band. From the latter property, they can be called *Möbius Kaleidocycles*.

Figure 1: *A collection of classical and Möbius kaleidocycles made from paper.*

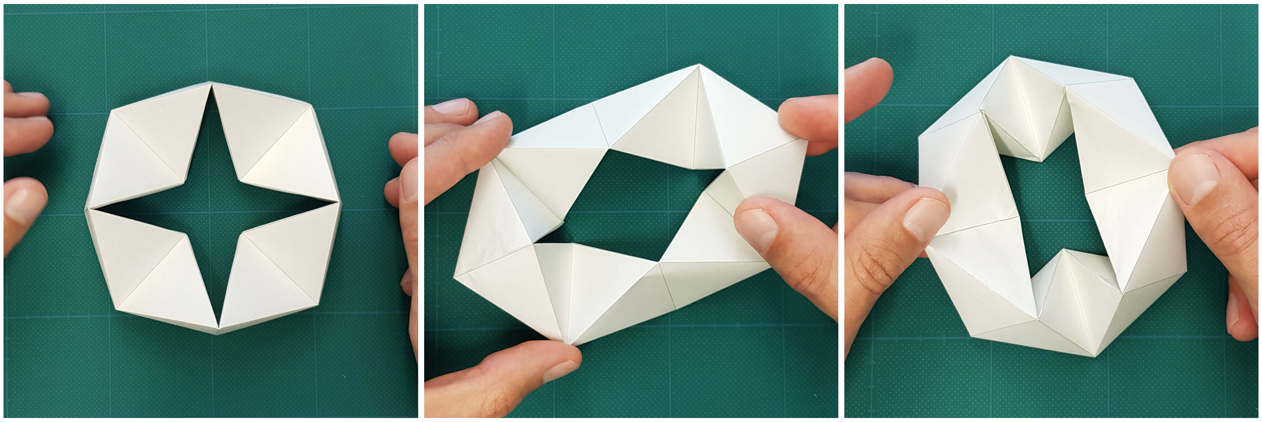

Figure 2: *A classical eightfold kaleidocycle has two independent degrees of freedom.*

Classical Kaleidocycles

An online search for the term “kaleidocycle” (which is greek and means “beautiful ring”) clearly demonstrates the popularity of these objects. The variety of different designs is overwhelming. For a classic reference see, for example, Cundy and Rollett [1]. Figure 1 shows a collection of kaleidocycles made in our lab. Kaleidocycles with M.C. Escher’s famous tesselations printed on their faces are a strikingly beautiful combination of mathematics and art. The book by Schattschneider and Walker [2] contains many such designs and their paper plans. For those who have never seen a kaleidocycle in action we suggest a short video [3].

The great majority of kaleidocycles now shown online are sixfold; they are made from six tetrahedra where two opposing edges of each tetrahedron serve as hinges which connect it to its neighbors. From the engineering point of view, kaleidocycles are mechanisms, the sixfold one being a member of the general family of 6R Bricard [4] linkages. Here, the “R” stands for *revolute* joint. Eightfold kaleidocycles are made from eight tetrahedra in the same fashion as the sixfold ones, but they are found much more rarely and there is a good reason for that. As can be seen in Figure 2 the eightfold cycle has two internal degrees of freedom, in contrast to the single degree of freedom of the sixfold cycle, and thus can be deformed in two independent directions at any instant. That makes the eightfold cycle more arbitrary and the situation becomes worse as the number of tetrahedral elements increases further. Formally, a closed ring of N tetrahedra linked by hinges has $N - 6$ internal degrees of freedom. This formula can be easily derived by noting that if viewed as free points in three space dimensions the $2N$ corners of the tetrahedra have $3 \cdot 2N$ degrees of freedom. But there are constraints in form of the tetrahedral edges which force pairs of corners to stay at fixed distance. The N hinges are such edges and each of the N tetrahedra provides four more such edges. So we end up with $5N$ constraints. Finally it is necessary to subtract the 6 degrees of freedom related to translations and rotations of a rigid body if the only interest is in the internal degrees of freedom, which then amount to $6N - 5N - 6 = N - 6$. This raises the question of why the sixfold cycle ($N = 6$) can move at all? This object is an example of an overconstrained mechanism; due to its high symmetry it possesses a hidden degree of freedom, as Bricard [4] explained. For more theoretical insights regarding symmetry and mobility of the sixfold cycle see Fowler and Guest [5]. Finally, no treatise about classical kaleidocycles should go without a reference to the inventor, artist, and industrial designer Paul Schatz (see the website of the Paul Schatz Foundation [6]).

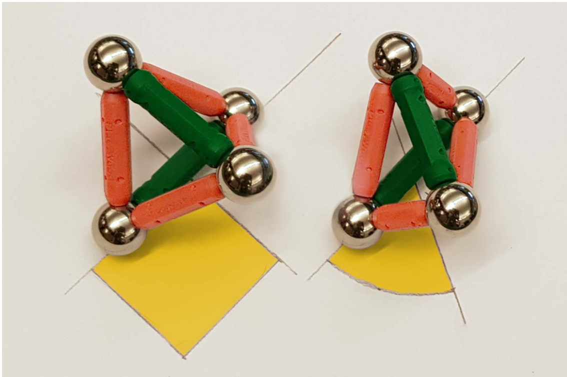

Figure 3: *Visualization of the twist angle for twisted tetrahedra. Left: In a regular tetrahedron the hinges (dark green) are orthogonal. Right: In the twisted tetrahedron the hinges (dark green) make an angle of less than* 90°*; in the shown case the angle is* 60° *(indicated in yellow).*

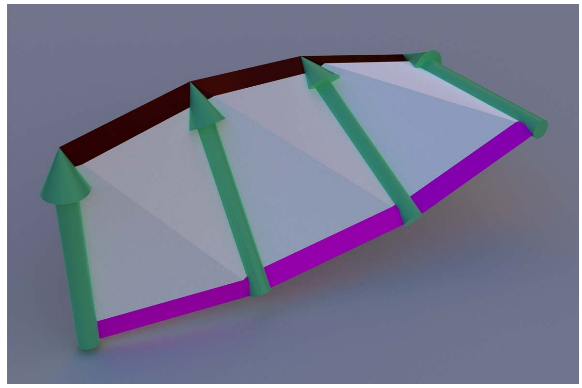

Figure 4: *Defining a "band" for a chain made from twisted tetrahedra. The hinges (green) are identified with vectors. The two edges of the band are defined by the tetrahedral edges connecting the origins (magenta) and the endpoints (dark brown) of the hinge vectors.*

Möbius Kaleidocycles Made From Twisted Tetrahedra

As of February 2018 an online search for kaleidocycles with an odd number of tetrahedra will yield very few hits. Is it even possible to make a closed ring from, say, seven tetrahedra? In fact it is possible, but if the two hinges on each tetrahedron are orthogonal — which is the case for all classical kaleidocycles discussed above — then such a sevenfold cycle must be made from very elongated tetrahedra and its shape and motion will not be very satisfying. However, we can generalize the shape of the tetrahedra in a way that allows the construction of highly intriguing rings. For this we consider disphenoids, which are "twisted tetrahedra" whose opposing edges have the same length. Under these circumstances the two hinges of each tetrahedron are generally no longer orthogonal. We refer to the angle formed by the hinges as the "twist angle" and denote it by α (Figure 3).

For a chain of twisted tetrahedra it is quite natural to assign an orientation to the individual tetrahedral elements, thereby allowing the chain to be viewed as a band with two "edges" and two "sides". For this we associate each hinge with a vector originating from one end of the hinge and pointing to the other end. The orientations for two hinge vectors shared by a tetrahedron are chosen such that the angle between them is less than 90°; in fact, this angle is the twist angle defined above. The two edges of the band are identified with the polygonal line segments connecting either the origins (first edge) or the endpoints (second edge) of the vectors, as indicated in Figure 4. For smaller twist angles this definition becomes increasingly obvious. Equipped with a sense of orientation the closure of the chain induces a topology for the ring. There are two ways to close the chain when bringing the terminal hinges together: The corresponding vectors are either parallel and the closed band is orientable or antiparallel and the band is nonorientable. Here we consider the nonorientable case which corresponds to the topology of a Möbius band with only one side and one edge.

For a nonorientable ring of N twisted tetrahedra there exists a critical minimal twist angle $\alpha_c(N)$ below which the ring cannot be closed. At this critical twist angle something surprising happens: The $N-6$ internal degrees of freedom which the ring generally possesses reduce to a single degree of freedom! The fascinating property of the classical sixfold kaleidocycle, namely that it has only one degree of freedom, thus extends to a ring with any number $N > 6$ of twisted tetrahedra with critical twist angle $\alpha_c(N)$. The angle $\alpha_c(N)$ can generally be obtained only numerically. The details of the computation are too involved to be shown here and

Table 1: *Critical twist angle α_c for a ring of N twisted tetrahedra.*

N	6	7	8	9	10	11	12	13	14	15
α_c	90°	72.819°	61.968°	54.183°	48.240°	43.525°	39.678°	36.473°	33.758°	31.427°

will be the subject of a forthcoming publication. In Table 1 we give numerical values of $\alpha_c(N)$ for $N \leq 15$. Since the topology of our critically twisted kaleidocycles is the one of a threefold Möbius band with three half turns, we call these new objects "Möbius Kaleidocycles".

For a Möbius Kaleidocycle with N elements the corresponding critical twist angle is a fundamental quantity which only depends on N. Mathematically, the critical twist angle is derived only from topological and orientational constraints. Specifically, these constraints require that the ring is closed and that neighboring hinges maintain a fixed twist angle. But if we consider a realization of the kaleidocycle as a physical object with linked tetrahedral bodies, the hinge length is a geometric design parameter governing the size of the tetrahedra. The hinge length is limited by the constraint that the tetrahedra must not overlap during any phase of the motion. Therefore, every Möbius Kaleidocycle has a specific maximum hinge length.

Although the classical sixfold kaleidocycle is sometimes erroneously termed a "hexaflexagon", there are actually some commonalities between these objects. Classical hexaflexagons are made from strips of flexibly connected equilateral triangles which are folded to form a layered hexagonal shape. Through a series of folding operations different sets of triangles will appear on the surface of the hexagon. For an entertaining introduction to hexaflexagons we suggest the videos by Vi Hart [7]; more theoretical background is provided, for example, in the book by Pook [8]. What kaleidocycles and hexaflexagons have certainly in common is that they are mechanisms which consist of rigid bodies — tetrahedra in the one case, triangular plates in the other — connected through hinges. Further, both objects allow for an everting motion and show different surfaces in different phases. Other aspects of hexaflexagons, like the discrete folding operations and the resulting combinatorial questions, are certainly distinct.

What We Will Do in this Workshop

Seeing a picture or even a video of a Möbius Kaleidocycle does not do justice to these objects. To hold one of them and observe its fascinating, highly nontrivial motion is a remarkable experience. Beyond that, considering the endless number of possible surface designs, every Möbius Kaleidocycle is a sculptural artwork in itself. For these reasons we will build different Möbius Kaleidocycle paper models in this workshop. Since the classically known kaleidocycles all have even numbers of elements we will focus on two oddfold Möbius Kaleidocycles, the sevenfold ($K7$) and the ninefold ($K9$) examples, and use each of these to elucidate different mathematical facts. The templates to all kaleidocycles discussed here and several other designs as well as the instructions for building them are available as supplementary pdf files.

Sevenfold Möbius Kaleidocycles

A $K7$ is a perplexing object which does not exhibit symmetry in any of its configurations. In fact, on viewing a $K7$ in Figure 5 for the first time it is difficult to believe that it can move at all! One reason for this is that we have maximized the hinge lengths for the particular design shown in that figure: If the hinges were any longer neighboring tetrahedra would overlap during motion. Additionally, each of the seven tetrahedra is oriented differently at any instant during the motion, which makes it at first challenging to envision how the object should move as a whole. The Möbius topology is also reflected by our $K7$ design. A colored stripe runs over its "one and only" side with numbers from 1 to 28, identifying all 28 triangular faces of the ring. During the everting motion of this $K7$ three different phases of the sequence of the first 28 natural numbers appear indefinitely because there is only one colored stripe which connects its ends (Figures 5 and 6).

Ninefold Möbius Kaleidocycles

In contrast to a $K7$, a $K9$ exhibits a threefold symmetry (Figures 8 and 9) that is shared by all Möbius Kaleidocycles whose numbers of elements N are divisible by three. It is already known that the classical sixfold kaleidocycle exhibits this symmetry. Different surface designs naturally lead to very different appearances. While the three color step design in Figure 8 results in a quite abstract icon-like object, the smoother rainbow color sequence in Figure 9 creates diffuse color clouds that traverse the whole color spectrum during the everting motion. What both designs have in common is that the color pattern is periodically repeated three times along the "one and only" side of the Möbius Kaleidocycle. If a different pattern is chosen, for example, only a single rainbow sequence instead of three, then the appearance is completely different. At the workshop we will explore several different surface design options. Further, we will provide plain white paper models which can be designed individually with markers or other means. The photographs in Figure 8 give a first idea of how the construction process during the workshop will proceed.

An Online Interactive Visualization Tool

Another way to discover and explore Möbius Kaleidocycles is via an online interactive visualization tool [9] that we have implemented (Figure 7). With this tool it is possible to: Select the number of tetrahedra; control the three-dimensional view; set the object into its everting motion with controllable speed; hide tetrahedra to inspect the individual motion of only a chosen subset of elements; change the hinge lengths; show trajectories traced by the tetrahedral corners or the hinge midpoints during motion; change the surface color design, and so on. Combining hands-on paper model building with this interactive visualization tool will further enhance the understanding of the underlying mathematical concepts.

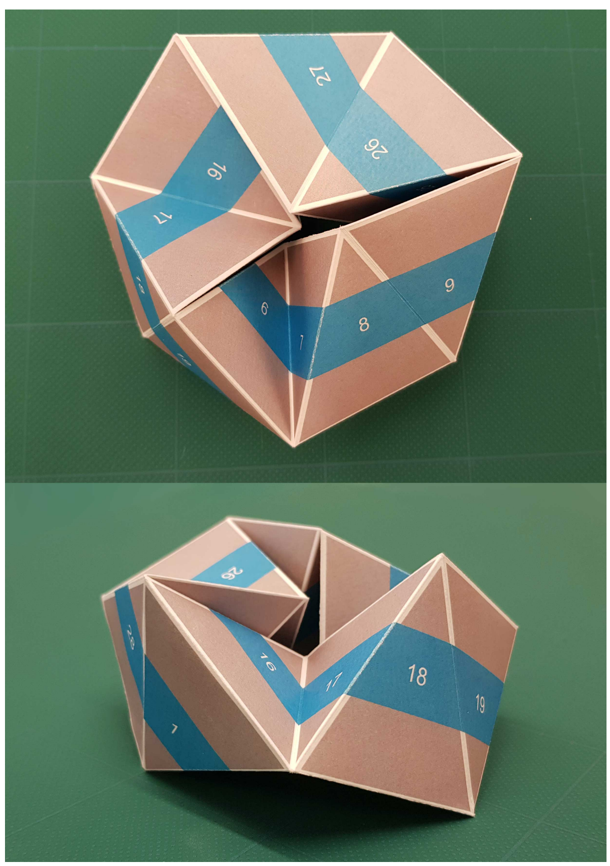

Figure 5: *Two different perspectives of a sevenfold Möbius Kaleidocycle ($K7$).*

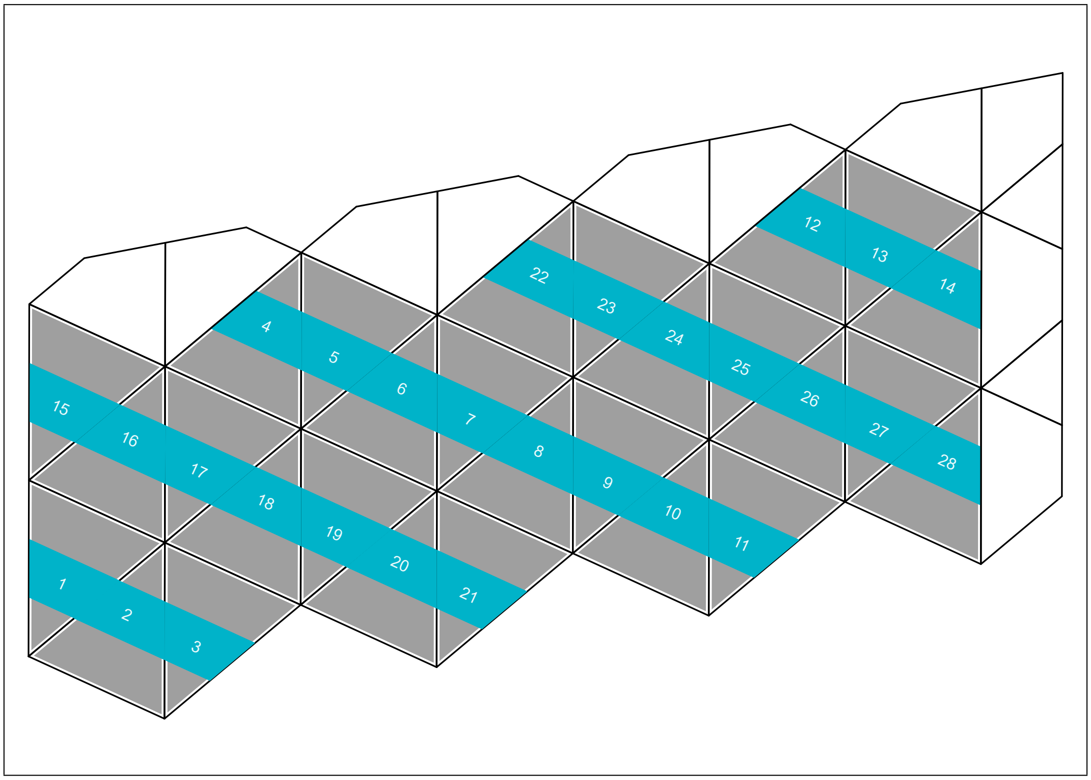

Figure 6: *Paper plan for the sevenfold Möbius Kaleidocycle (K7) in Figure 5.*

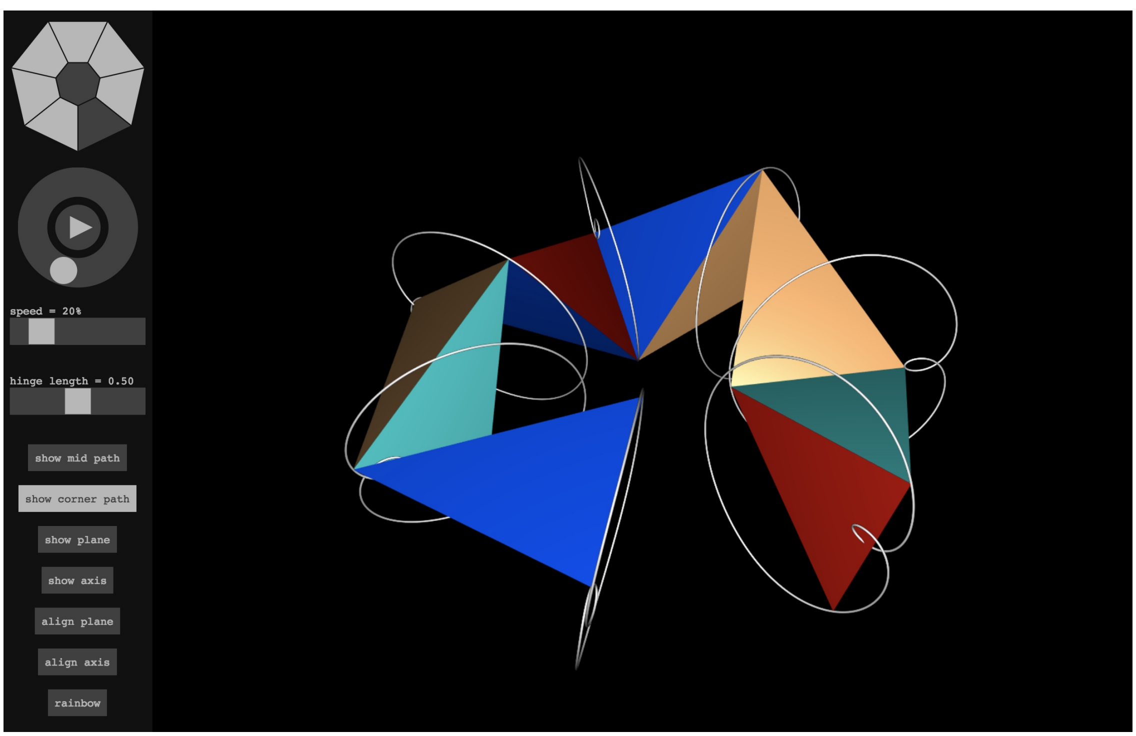

Figure 7: *An online interactive visualization tool for Möbius Kaleidocycles.*

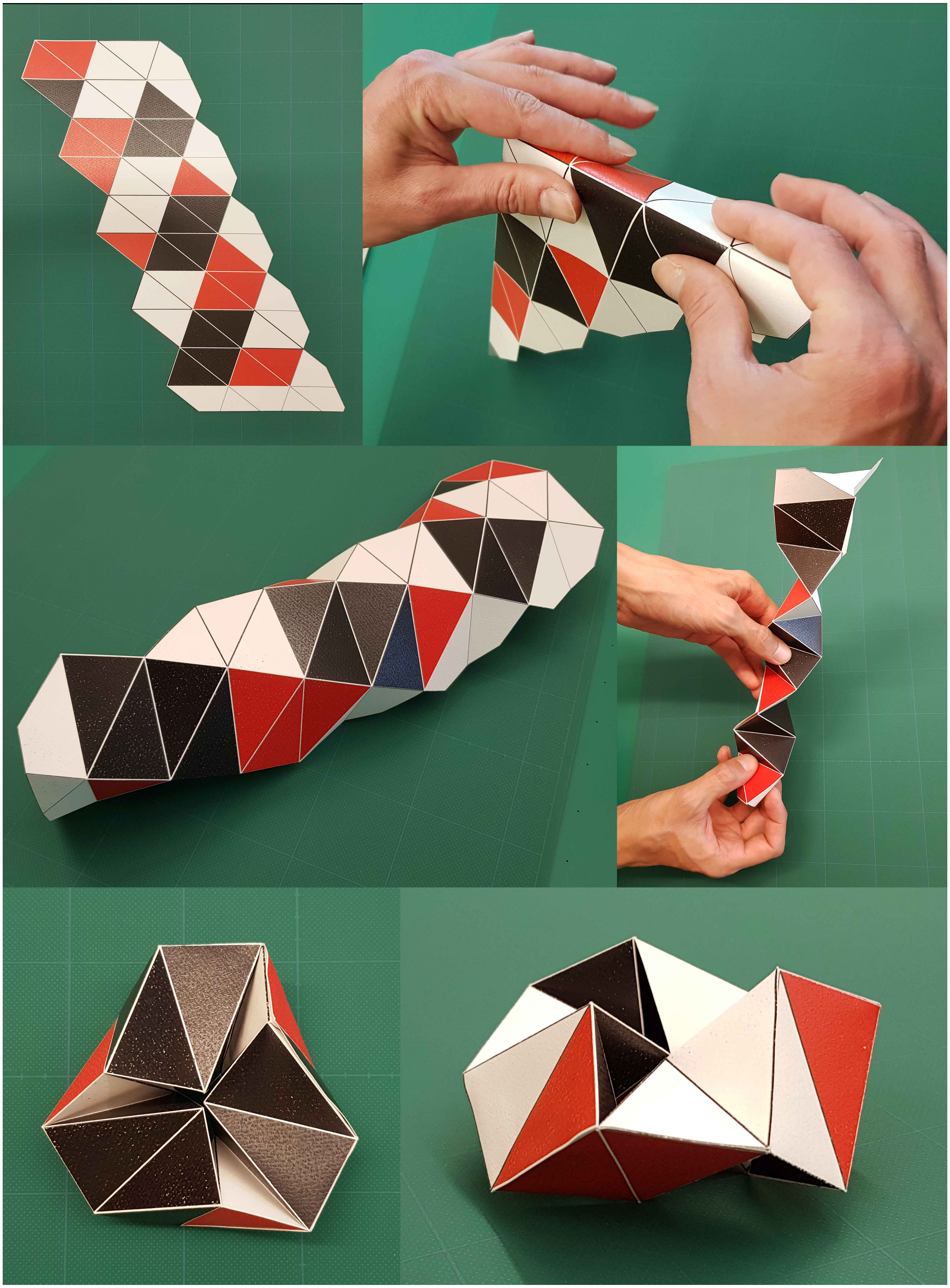

Figure 8: *Paper plan, construction process, and final result for a ninefold Möbius Kaleidocycle ($K9$).*

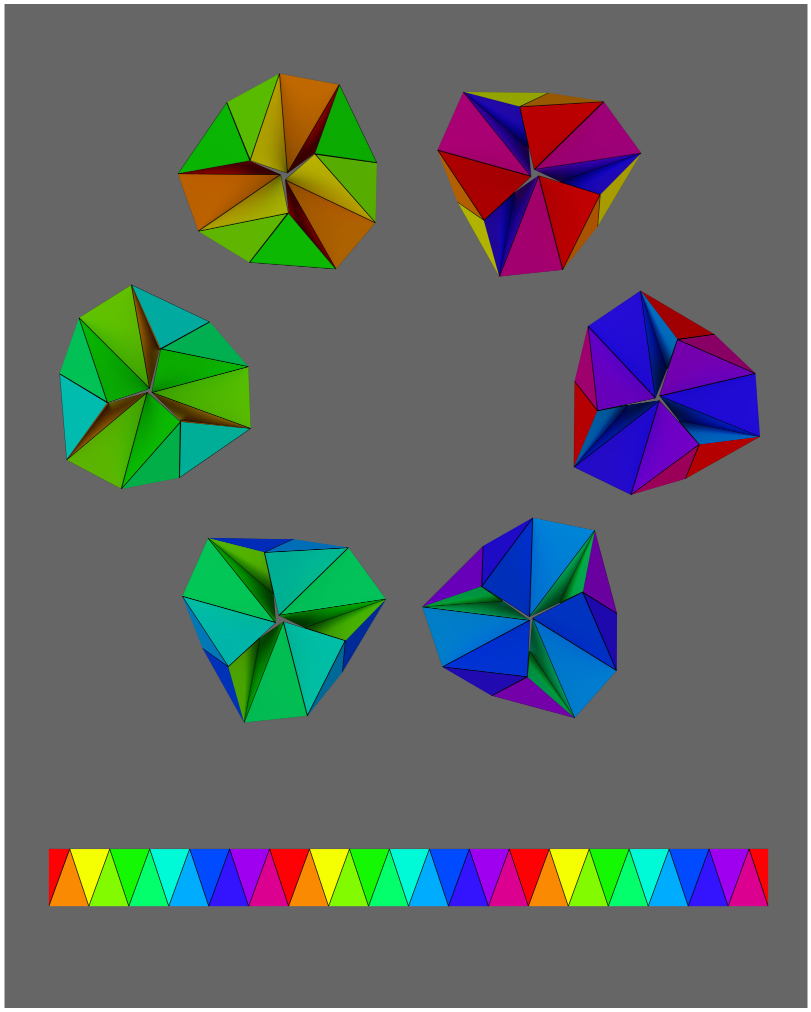

Figure 9: *A ninefold Möbius Kaleidocycle ($K9$) with an alternative rainbow color design. The top shows six different phases during the everting motion. The stripe design at the bottom illustrates that the periodic rainbow spectrum is applied for three full periods.*

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