Folding the Vesica Piscis

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Abstract

Inspired by a sculpture by Susan Latham, we study a two-parameter family of shapes obtained by folding two overlapping circles, so that their boundaries coincide and the involved surfaces are cylinders or cones. Through determination of explicit parametrizations, we prove their geometric existence. Furthermore, we find a one-parameter subfamily, where the creases are circular in their development.

Introduction

The so-called *Vesica Piscis* consists of two intersecting congruent circles, so that the centers lie on the resp. other circle. It is not surprising that this simple formation has found different applications throughout history, for example, the construction of an equilateral triangle in Euclid's Elements, see Figure 1.

Our interest in this figure was sparked by the sculpture "Attraction" by the artist Susan Latham, where two Vesica Piscae are cut out of sheet bronze and each folded along the inner circle segments, so that the outer segments coincide, see Figure 2. Although the manual generation of this shape is fairly simple, the geometric reconstruction is not supported in common CAD software. This initiated a collaboration between a designer and a mathematician, with the common interest to explore the design space of closed shapes that are obtained by perimeter gluing of developable surfaces with curved creases.

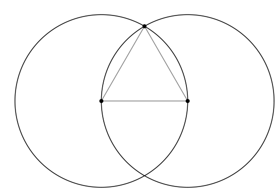

Figure 1: *Vesica Piscis in Euclid's Elements*

The topic of closed shapes bounded by developable surfaces, originates from D-Forms, which were introduced by Wills in [6] and formalized by Pottmann and Wallner in [5]. Demaine and Price generalize these shapes in [1] to so-called *seam-forms*, which are convex bodies that under certain conditions allow also gluing multiple regions. Moreover, they prove that seam-forms are the convex hull of their seam-curves and that creases can only occur between vertices of the boundary curves or are tangent to them. This implies that D-Forms are, except for their boundary curves, crease free.

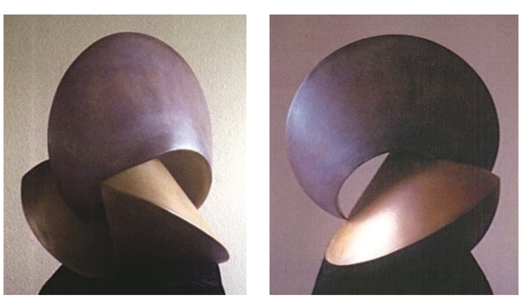

Figure 2: *Susan Latham's "Attraction", Santa Fe (2008)*

However, the problem with paper folding is that the material appears to be 'forgiving' as it can be folded into shapes that cannot be said to exist in the mathematical world. One example of a non existing object, namely the pleated hyperbolic paraboloid, is given in [2]. Thus it was firstly not self-evident that the folded Vesica Piscis actually 'exists' and that it's developed crease curves are circular. Furthermore, it is a non-trivial representative of, as far as we know, unexplored family of closed shapes bounded by developable surfaces with curved, non planar creases that are glued along the surface boundaries.

We started the analysis of the folded Vesica Piscis by numerical experiments, which led us to assume that the object consists of cylindrical and conical surface patches. In the following, we verify this by determining explicit parametrizations. Moreover, we generalize this approach to developments consisting of two overlapping discs, which we pursue to obtain a two-parameter family of closed shapes. The benefit of our this approach are the explicit parametrizations, that also can be used for the computation of developed crease curves for the shapes made from two overlapping circles folded into a cylinder and two cones.

Two-Parameter Family of Folded Closed Shapes from the Union of Two Discs

In the following, we investigate the shapes consisting of one cylindrical and two conical surface patches, that are isometric to the union D^u of two discs of radius 1 and centers $(\pm u, 0, 0)$ located in the xy-plane. Assuming that the seam curve remains planar and the developed cones' vertices coincide with the origin, we determine the parametrization of the seam and crease curves. Due to the symmetry of this shape, we consider only the half of the region D^u, which is bounded by the y-axis and the circle

$$\bar{c}_u(s) = (\cos s + u, \sin s, 0) \quad \text{with} \quad s \in [s_0, s_1] = [-\arccos(-u), \arccos(-u)].$$

The Vesica Piscis corresponds to the special case of $u = \frac{1}{2}$.

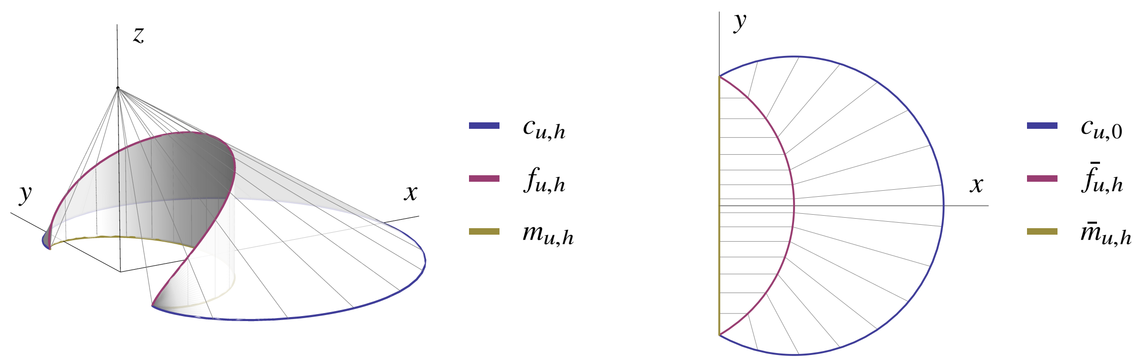

Figure 3: *Parametric setup for the folded and initial state of* $D^u|_{x\geq 0}$

Seam Curve

We firstly determine the cone with apex $v_h = (0, 0, h)$ and planar boundary curve $c_{u,h}$, which development corresponds to the given half of D^u. We parametrize $c_{u,h}$ in polar coordinates, i.e.

$$c_{u,h} = r_{u,h}(\cos \alpha_{u,h}, \sin \alpha_{u,h}, 0) \quad \text{and} \quad c_{u,0} = \bar{c}_u.$$

As isometry preserves the lengths of the cones' rulings and the arc-length parametrization of the boundary curve, we deduce

$$r_{u,h} = \sqrt{\bar{c}_u^2 - h^2} = \sqrt{2\cos u + u^2 + 1 - h^2} \tag{1}$$

and

$$|c'_{u,h}| = \sqrt{r'^2_{u,h} + r^2_{u,h}\alpha'^2_{u,h}} = 1 = |\bar{c}'_u| \quad \Longrightarrow \quad \alpha'^2_{u,h} = \frac{1}{r^2_{u,h}}\left(1 - r'^2_{u,h}\right).$$

Inserting (1) and simplification yields, that $\alpha_{u,h}$ is up to the orientation of parametrization and rotation determined by

$$\alpha'_{u,h} = \frac{\sqrt{(u\cos s + h + 1)(u\cos s - h + 1)}}{2u\cos s + u^2 - h^2 + 1}.$$

The integral is real-valued for $0 \leq h \leq h_{\max}(u) = 1 - u^2$ and can be written as a linear combination of elliptic integrals of the third kind[1], namely

$$\alpha_{u,h}(s) = \frac{1}{2}\left(\gamma_0 \Pi(\delta_0, \phi(s), m) + \gamma_1 \Pi(\delta_1, \phi(s), m)\right),$$

where the Jacobi amplitude and elliptic modulus are given by

$$\phi(s) = \arctan\left(\sqrt{\frac{1-u+h}{1+u+h}}\tan\frac{s}{2}\right) \qquad \text{and} \qquad m = \frac{4uh}{1-(u-h)^2}.$$

The elliptic characteristics and coefficients read

$$\delta_0 = \frac{-2u}{1-u+h}, \qquad \delta_1 = \frac{2u}{1+u-h}, \qquad \gamma_0 = \frac{2(1+u+h)}{\sqrt{1-(u-h)^2}}, \qquad \gamma_1 = \gamma_0\frac{1-u-h}{1+u+h}.$$

Refer to [4] for the derivation of generalized integrals.

Crease Curve

The crease curves representation f_h and its development $\bar{f}_h$ on the cone reads

$$f_{u,h} = (1-t_{u,h})v_h + t_{u,h}c_{u,h} \quad \text{and} \quad \bar{f}_{u,h} = t_{u,h}c_{u,0} \quad \text{with} \quad t_{u,h} = \frac{h}{\cos s + u + h}. \tag{2}$$

Figure 4 illustrates the behaviour of the herby computed developed crease curves, which depend on the parameters u and h that depict the circle overlap and the cones vertex height $h \leq h_{\max}(u)$ resp.

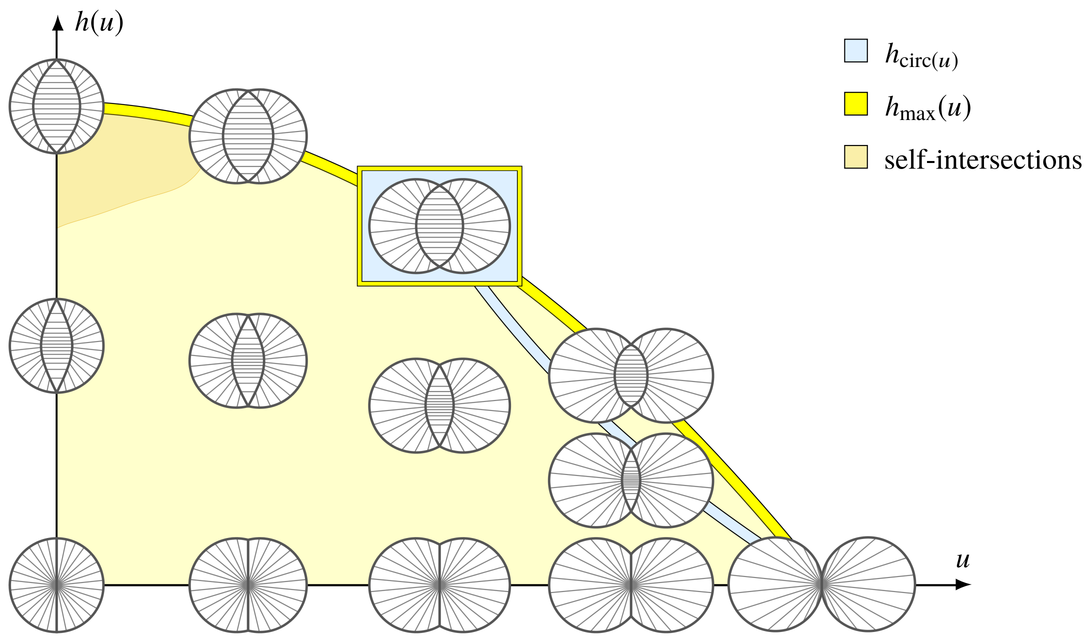

Figure 4: *Diagram representing the crease curves of the two-parameter family of closed shapes. It follows from numerical computations that $\alpha_{u,h}(s_1)$ exceeds π for $u \leq 0.19$ and high values of h. The corresponding folded shapes exhibit self-intersections.*

[1] $\Pi(l;\phi,k) = \int_0^\phi \frac{1}{(1-l\sin^2\theta)\sqrt{1-k\sin^2\theta}}\,d\theta$

Equation (2) follows from the requirement that the z-value of $f_{u,h}$ equals the x-value of $\bar{f}_{u,h}$. Furthermore, it can be verified that the parametrizations of the midline $m_{u,h}$ obtained by tracing the rulings of the composed developed surface starting from the arc length parameter of $c_{u,h}$ corresponds to the parametrization of its development $\bar{m}_{u,h}$. The resulting crease curves are in general not planar. Note, that Equation (2) holds for any family of cones with planar boundaries $c_{u,h}$ and vertices v_h. For further implications we refer to [3].

Circular Crease Curves

Surprisingly, the creases are circular for heights of the cones vertices

$$h_{\text{circ}}(u) = \frac{1-u^2}{2u} \quad \text{for} \quad u \in \left[\frac{1}{2}, 1\right].$$

The original folded Vesica Piscis is the special case of $u = \frac{1}{2}$ and $h = h_{\text{circ}}\left(\frac{1}{2}\right) = h_{\max}\left(\frac{1}{2}\right) = \frac{3}{4}$.

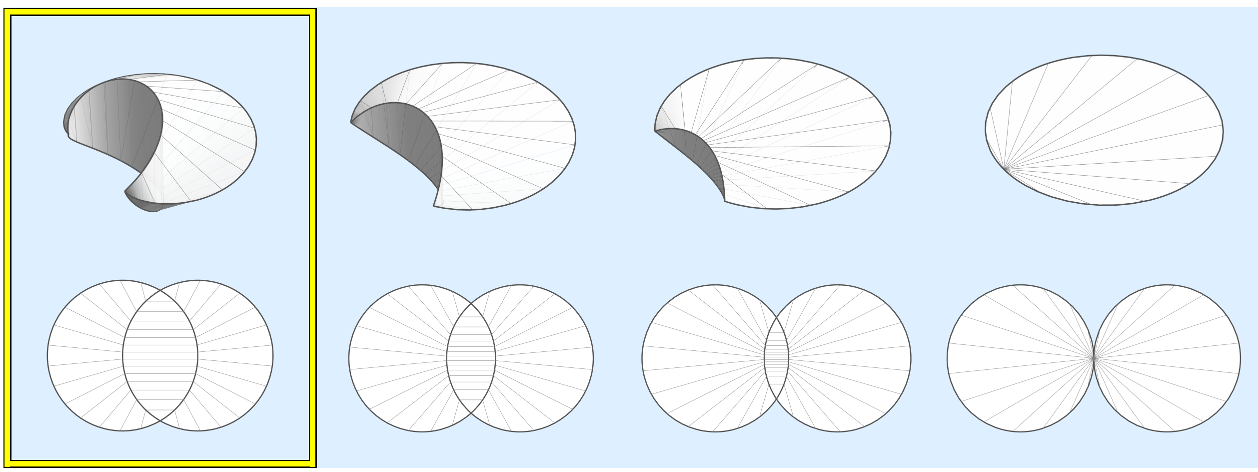

Figure 5: *Examples of folded closed shapes with circular developed creases*

Acknowledgements

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