A Plethora of Patterns from Discrete Spiraled Sequences

Tom Bates

Santa Barbara, California, USA; tombagoo@gmail.com

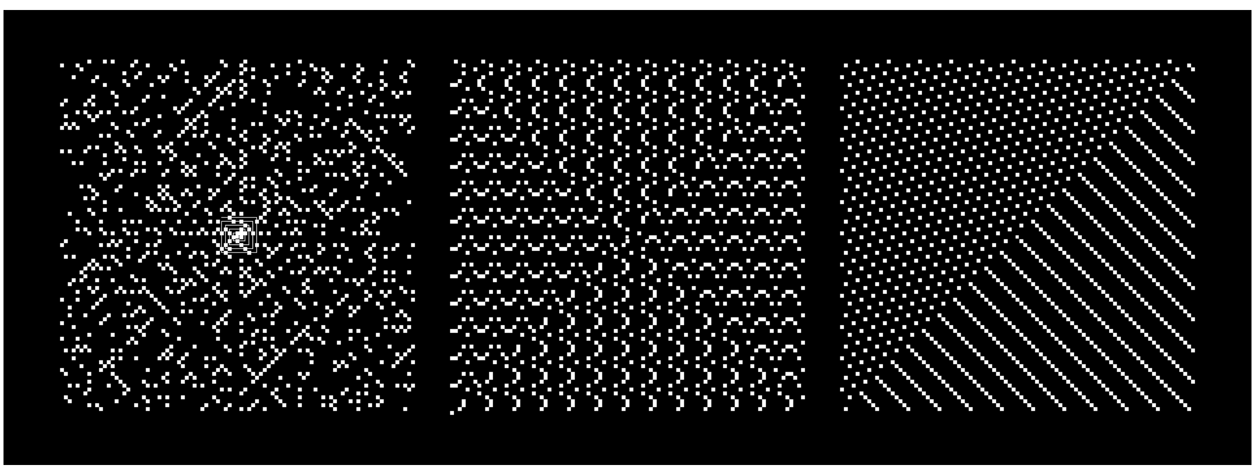
Abstract

Inspired by Stanislaw Ulam's spiral of prime numbers on a square grid, a similar approach is used on more mundane integer sequences, with visually surprising results. Starting from single integer multiples displayed on both square and hexagonal grids, which give rise to visually interesting patterns for every positive integer, a simple parameterized indexing algorithm is then used to create a profusion of patterns "in between" the integer multiple ones. This work can serve as the basis for artistic images and animations, and is presented at a level that should be accessible, and perhaps inspirational, to a wide range of STEAM students.

Inspiration

Some of the first computer graphics were created by the mathematician Stanislaw Ulam. Using the Maniac II computer at Los Alamos Scientific Laboratory in the 1960s, he created a few images by marking the primes on a number line curled up into a square spiral, reproducing a doodle he had made while at a conference. He was intrigued by a "diagonal dust" that appears in such an image. Ulam and two colleagues wrote a paper about this, attributing the dust to the known propensity of some quadratic equations to yield a fair number of primes [1].

While reproducing Ulam's prime spiral images for fun, and in particular while making an animation of the sieve of Eratosthenes, I became intrigued by the patterns made just by marking the multiples of any given integer on such a grid. Ulam's spiral, along with the multiples of 7 and 8, are shown in Figure 1.

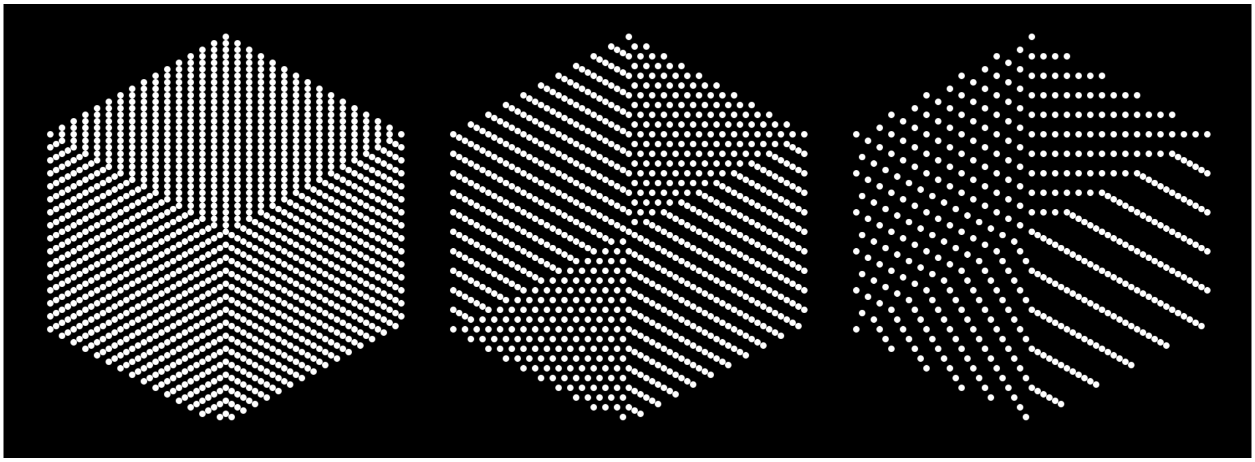


(a) (b) (c)

Figure 1: *Square spiral grid patterns. a)Ulam's prime numbers spiral. b) The multiples of 7. c) The multiples of 8.*

In Figure 1b we see a sort of arch-shaped "stamp" that is repeated, with 90 degree orientation changes at each turn of the spiral path. This is a common motif, with the details differing for each integer. Such stamps are contained in a square that is *n* x *n* cells large, where *n* is the the integer whose multiples are displayed. Some integers, however, especially but not exclusively powers of two, give rise to a diagonal bifurcation of the plane with each side containing a distinct pattern.

In Figure 2, marks are placed on a spiral hexagonal path. For both the square and hexagon, the points at which marks are placed are the center points of the tiles of square or hexagonal tessellations, with the point with index zero at the center.



(a) (b) (c)

Figure 2: *Hexagonal grids showing integer multiples: (a) of 2, (b) of 3, (c) of 6.*

Between Mere Multiples

As visually surprising as these images of pure multiples are, it turns out they merely bracket a large number of images that, in a very specific sense, lie in between them (I refer to these below as "inbetweeners"). There are so many of these in betweeners that one can create mesmerizing animations between any two integer multiple patterns. Unfortunately, animations cannot (yet) be included in a paper.

In the case of the multiples of some integer *n* we have the trivial mapping

$$f: X \rightarrow Y,\ f(x) = n * x, \tag{1}$$

where X is the set of integers (or indices) for every position on the integer number line, whether curled up into a spiral or not, and Y is then the set of multiples of *n*. Marks are then placed on the grid for each value in Y.

A simple example of an in-between mapping that can generate animations is to take

$$f: X \rightarrow Y,\ f(x) = \text{floor}[(n + t * \varepsilon) * x], \tag{2}$$

where t is a positive integer (e.g. the frame number) and ε is a positive value less than 1. The floor function takes the result to an integer, which is required since these polygonal spirals have only discrete positions. We can see even on the number line what gives rise to the inbetweeners.

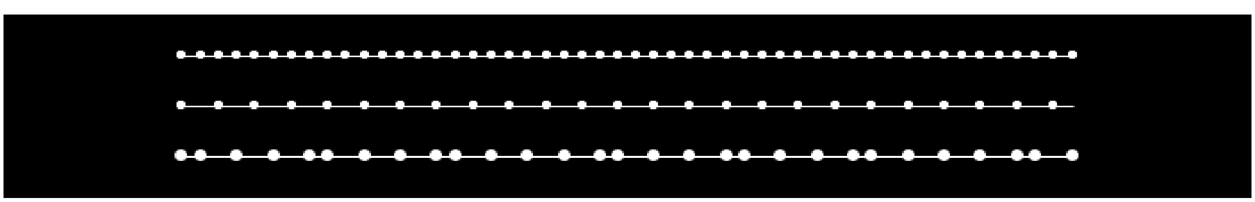


Figure 3: *Three number line markings. Top: multiples of 1. Middle: multiples of 2. Bottom: an inbetweener with n = 1, t = 77, and ε = 0.01, yielding a 1.77 mapping.*

The bottom line in Figure 3 mixes the gaps of the 1- and 2-multiple cases because the mapped-to value does not advance uniformly for each value of x, as in the pure multiple case, but only when $t * \varepsilon * x$ in (2) yields an integer. In fact, the gaps alternate in such a way that the ratio of the number of larger gaps to the total number of gaps approaches exactly the value of the fractional part of $t * \varepsilon$ in (2) if enough marks are made.

The number line is useful for illustrating what is being done, and even suggests musical beat pattern possibilities, but the real visual fun comes from in-between mappings displayed on a discrete spiral. The patterns that result are very sensitive to the values of t and ε, leading to a cornucopia of alignments and resonances as the product of these two values changes even slightly. Figures 4 and 5 show a few of these.

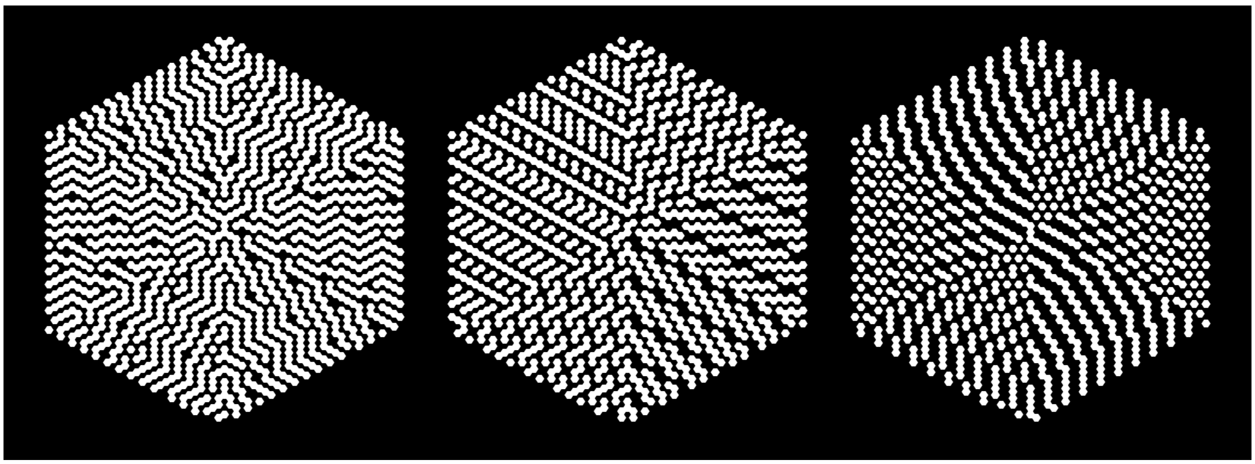


Figure 4: *Inbetweeners lying between the multiples of 2 and 3 as shown in Figure 2 (a) and (b). Here, in (a) the index multiplier is 2.1, in (b) 2.4, and in (c) 2.984.*

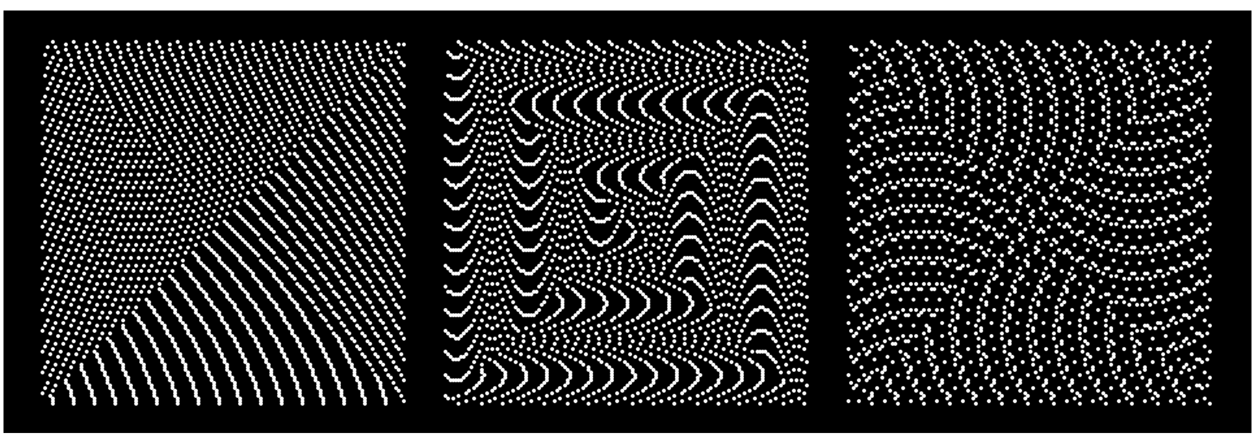


Figure 5: *Inbetweeners on the square spiral. The index multipliers are (a) 8.01, (b) 8.3, (c) 9.97. Notice that (a) is quite close to the pure multiple of 8 pattern in Figure 1c.*

Summary and Conclusions

I am interested in the artistic power of constraints. When I began exploring these patterns, I was curious to see what might come of the constraint that marks be placed only at discrete locations on a line, rather than marks that could slide around continuously. Two number lines, one marked with the multiples of n, and the other with the multiples of m, obviously show different, non-superimposable, regular patterns, yet they are just a scaling of n/m away from being superimposable. But once folded around themselves in some way, this is no longer the case. And when one dives in between the pure multiples, the patterns that appear as different parts of the number line are brought into coincidence with each other is, for me, a powerful yet simple source of surprising visual complexity. The two-dimensional patterns amplify the uniqueness of the one-dimensional ones, giving them a visual interest they do not possess in their linear state.

To me this is an illustration of the beauty that hides, waiting to be discovered, even in very simple mathematics, especially when an experimental and playful visual approach using computers is included in the process. As an occasional volunteer programming instructor, I see value in exploring relatively easy-to-code processes that can make STEAM students more aware of the deep artistic and creative potential of computers, and thus more willing to learn the hard or tedious bits of programming.

In this paper only a sketch of this topic has been given. I am currently using these ideas to create images and animations where the marks employ size changes, overlapping, semi-transparency, and other techniques, as in Figure 6.

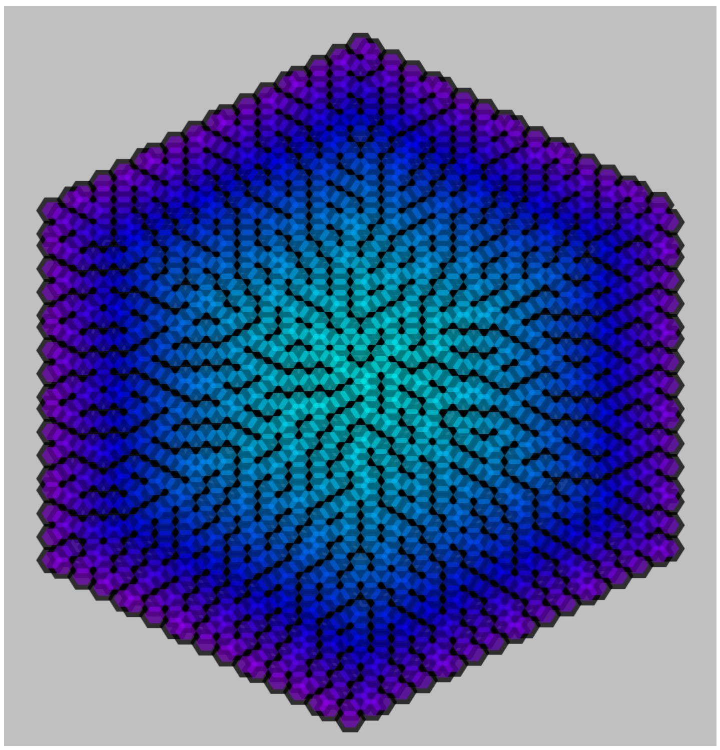


Figure 6: *One frame of an animation. This frame is based on the 1-2 inbetweener at 1.74.*

Acknowledgements

My thanks to Stanislaw Ulam for doodling while bored at a mathematics conference, and Maniac II.

References

[1] M. L. Stein, S. M. Ulam, and M. B. Wells. “A Visual Display of Some Properties of the Distribution of Primes.” *The American Mathematical Monthly*, vol. 71, no. 5, 1964, pp. 516–520.