Which Integer Is the Most Mysterious?

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Abstract

The author argues that the number 248, together with some of its multiples, is the Most Mysterious (large) Integer that is encountered in Mathematics, Physics, and Mysticism.

A Preliminary Discussion – and a Word of Warning

"All things are number," Pythagoras may have said. The Pythagoreans assigned special meanings to small integers: For example, 1 was the number of reason and unity; 2 represented the female principle; 3, the male principle; 4, the four elements; 5 = 2 + 3, marriage; etc.

Other philosophical traditions and religions have their sacred numbers, as well. Surely, the number 3 is important in Christian theology as it represents the Holy Trinity, whereas 5 is sacred in Islam as it corresponds to its Five Pillars.

So there would be many candidates for the Most Mysterious Number in the Universe that we are looking for in this paper. Let us exclude from discussion, however, the smallest integers because far too many different meanings can be assigned to them.

Among the bigger ones, 666, or the Number of the Beast, is famous enough but, in spite of two millennia of speculation, no conclusive mathematical interpretation has been found for it. So let us also exclude numbers whose only special meaning is religious or cultural.

Some physicists are obsessed [5] with the number 137 because the (dimensionless) fine-structure constant is approximately $1/137$. However, this is but an approximation, and no justification for a precise integer value has ever been found.

A long time ago, I wrote a humorous paper in the form of a ghost story [6] about some of the mysterious numbers and numerological coincidences that haunt fundamental science for reasons that we mainly still ignore. In what follows, we will show that the number 248, together with some of its multiples, holds a very special place in Mathematics and Physics, and surprisingly enough, also in Mysticism.

A word of warning is necessary: In this paper we discuss several advanced concepts at a meta level. We make no effort to define all the terms that we use. Indeed, that would require massive amounts of explanation far beyond the scope of a Bridges paper. The bulk of our subject matter comes from String theory. Precise references can be gleaned from any standard textbook of String theory, e.g. [2], or the classic [4].

The Number 248 in Mathematics

Sir Michael Atiyah, a Fields medalist of 1966, once stated in an interview [1]:

"For example, the classification of Lie groups is a bit peculiar. You have this list of groups, both classical and exceptional. But for most practical purposes, you just use the classical groups. The exceptional Lie groups are just there to show that the theory is a bit bigger; it is pretty rare that they ever turn up."

He got it wrong. Some of the exceptional Lie groups, E_8 in particular, *do* play a foundational role both in Mathematics and in Physics. Wilhelm Killing first discovered the exceptional Lie algebra E_8 in the 1880s, and Élie Cartan proved the existence of the corresponding Lie group E_8. The E_8 root system which contains 240 root vectors spanning $\mathbb{R}^8$ was first discovered by Thorold Gosset, an English amateur mathematician, in 1900. The exceptional Lie group E_8 thus has dimension $240 + 8 = 248$, and no other simple Lie group has that particular numerical property. The E_8 lattice (or Gosset lattice) generated by the E_8 root system in 8-space has the property that it realizes the optimal sphere packing in 8 dimensions.

The structure of E_8 has recently been illustrated by two different means that can be considered as sophisticated works of mathematical art. First, in January 2007, a team of twenty scholars of the Atlas of Lie Groups project, coordinated by the American Institute of Mathematics, announced having accomplished a certain huge computation elucidating the structure of E_8. In the ensuing media campaign, a picture of a 2 D projection of the E_8 root system became iconic. Secondly, several people have constructed Zometool models of partial 3 D projections of the same root system.

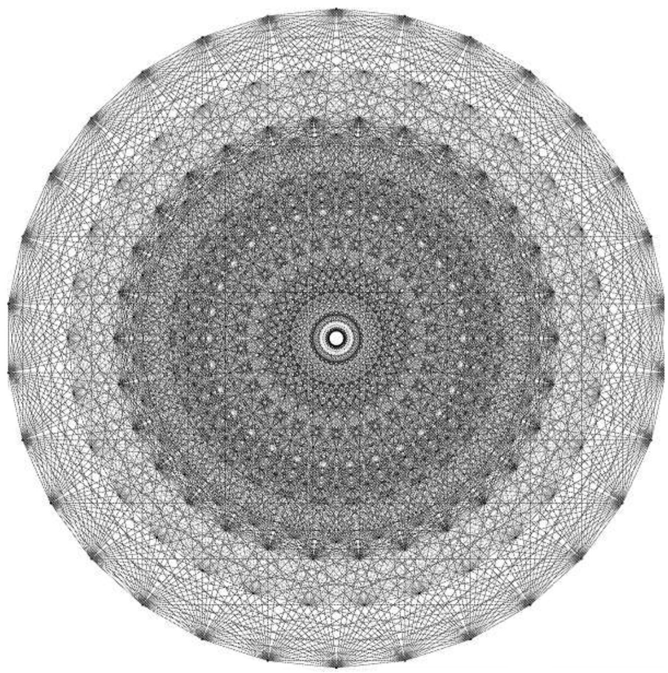

Figure 1: *A 2 D projection of the E_8 root system. Picture credit: American Institute of Mathematics.*

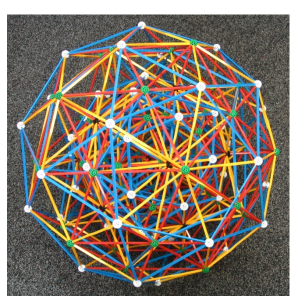

Figure 2: *A Zometool model of a partial 3 D projection of the E_8 root system. Picture credit: Wikimedia commons.*

The Numbers 248 and 496 in Physics

String theory is an attempt to describe all elementary particles, as well as their interactions, within a single theory. Starting from 1984–1985, several scholars realized that such a "Theory of Everything" (TOE) requires a background space of $D = 10$ dimensions and a gauge group of dimension 496. Indeed, Superstring theory (which also involves supersymmetry, i.e., a superpartner for every elementary particle) turns out to be anomaly-free, i.e., consistent, only under these very specific circumstances.

A glance at the classification table of real simple Lie groups shows that there are only two natural candidates for the gauge group: $E_8 \times E_8$ and $SO(32)$ which both have the required dimension $496 = 2 \times 248$. One speaks of heterotic $E_8 \times E_8$, and heterotic $SO(32)$, superstring theory, respectively. Their discovery which was made in 1985 by the so-called "Princeton String Quartet" (David Gross, Jeffrey Harvey, Emil Martinec, Ryan Rohm) heralded what has retrospectively been called the First Superstring Revolution.

The Second Superstring Revolution started around 1994–1995 when Edward Witten (a Fields medalist of 1990) and others described six different theories – the $E_8 \times E_8$ heterotic superstring, the $SO(32)$ heterotic superstring, three other string theories called Type I, Type IIA and Type IIB, and 11-dimensional supergravity – as different limits of a single theory that they called M-theory dwelling in a background space of $D = 11$.

The numbers 248 and 496 have thus become an inseparable part of fundamental physics.

744 and Monstrous Moonshine

The third multiple of 248, or 744, also plays a peculiar role in Mathematics. Consider Klein's j-invariant, or the j modular function, which regarded as a function of a complex variable τ is a modular function of weight zero for $SL(2,\mathbb{Z})$. The q-expansion of the j function, or its Laurent series in terms of $q = e^{2\pi i \tau}$, begins as:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + 333202640600q^5 + \dots$$

In a pioneering paper of 1979, John H. Conway and Simon P. Norton [3] pointed out a strange phenomenon that they called "monstrous moonshine": The coefficients of the above expansion can be expressed as simple (non-unique) linear combinations of the dimensions of the irreducible representations r_n of the Fischer-Griess Monster group, the largest sporadic finite simple group having order $808{,}017{,}424{,}794{,}512{,}875{,}886{,}459{,}904{,}961{,}710{,}757{,}005{,}754{,}368{,}000{,}000{,}000 \approx 8 \times 10^{53}$. Explicitly,

$$\begin{aligned}
1 &= r_1\\
196884 &= r_1 + r_2\\
21493760 &= r_1 + r_2 + r_3\\
864299970 &= 2r_1 + 2r_2 + r_3 + r_4\\
20245856256 &= 3r_1 + 3r_2 + r_3 + 2r_4 + r_5\\
&= 2r_1 + 3r_2 + 2r_3 + r_4 + r_6\\
333202640600 &= 5r_1 + 5r_2 + 2r_3 + 3r_4 + 2r_5 + r_7\\
&= 4r_1 + 5r_2 + 3r_3 + 2r_4 + r_5 + r_6 + r_7; \text{ etc.}
\end{aligned}$$

The mystery of monstrous moonshine was elucidated by Richard Borcherds, a Fields medalist of 1998, who showed that lying behind the phenomenon – as earlier conjectured by Igor Frenkel, James Lepowsky, and Arne Meurman – is the so-called "monster vertex algebra" which has the Monster group as its group of symmetries. Borcherds used in his proof techniques and tools from String theory, thereby bridging in a deep, and still not thoroughly understood manner several areas of Mathematics and Physics. As for the constant term of the above j expansion, it could be, in principle, arbitrarily changed. Even so, the appearance of $744 = 3 \times \dim(E_8)$ in this context might be more than an accident, but this is an open question.

The Number 992 as a Constraint of Supergravity

Next we will see that the fourth multiple of 248, or 992, also plays a privileged role at the junction of Mathematics and Physics.

It is a mathematical fact that some spheres S^n may carry a non-trivial number of non-diffeomorphic differential structures. The existence of such *exotic spheres*, i.e., non-trivial differential structures on spheres, came as a surprise when John Milnor (a Fields medalist of 1962) constructed the first examples on S^7 in 1956. The total number e_n of distinct differential structures is known for spheres of dimension ≤ 63 (except for $n = 4$ where the problem remains open). For instance, $e_7 = 28$, and it is a fact of life that $e_{10} = 992$. Now consider the fact that the real dimension of complexified $E_8 \times E_8$ (resp. $SO(32)$) is $2 \times 496 = 992$. For gravity, the group of diffeomorphisms Diff plays the role of the gauge group. The fact that Diff(S^{10}) breaks into 992 components amounts to a differential topological reason for the anomaly cancellation of the supergravity sector of M-theory in 11-dimensional space where S^{10} is naturally embedded.

By now the reader should be convinced that the number 248, together with its multiples 496, 744 and 992, bridges fundamental Mathematics and Physics in several seemingly independent ways. Our understanding of M-theory, let alone TOE, remains limited. A "Third Superstring Revolution" would be needed to further elucidate these difficult issues.

The Number 248 in Ancient Mystic Traditions

No discussion of the number 248 would be complete without mentioning its prominence in Jewish theology.

It is well-known that Hebrew letters have numerical values which has given rise to Gematria, the art of assigning numerical values to words of the Scripture, and thereby mystical meanings to numbers. As for 248, it is, first of all, the gematric value for the Hebrew letters 'Ramach', traditionally depicted as the number of bones in the human body. It is also the number of positive commandments in the Torah, and the number of words of the prayer Shema Yisrael. These are ancient and well-established statements that, inevitably, have given rise to much modern mysticism, given the newly-discovered prominence of 248 in fundamental physics. The ubiquity of 248 in M-theory thus should reflect – according to some Jewish religious thinkers – a supposedly deep numerological correspondence between Cosmos and Man, Macrocosm and Microcosm.

Let us give the final word to Pythagoras. Did he have any hunch of the critical dimension $D = 10$ of Superstring theory and of the anomaly cancellation for its critical gauge group dimension 496?

Well, Pythagoras taught his disciples two and a half millennia ago that the holiest number among the small integers was the Tetractys, or 10, because it is the *triangular number* related to the Four Elements, $10 = 1 + 2 + 3 + 4$. Moreover, he also knew that 496 is a *perfect number*, i.e., a number that can be represented as the sum of its dividers: $496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$.

References

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