4-3 Dissection Tiling System

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Abstract

This paper defines a new type of system related to mathematical dissections, called a dissection tiling system. In this system, base shapes of a geometric dissection must tessellate, individual components must tessellate, and all combinations of individual components must tessellate. Further exploration focuses on the pattern variations made possible by the 4-3 dissection tiling system.

Introduction

Mathematical dissections have two related concepts, geometric dissections and dissection tilings. A *geometric dissection* is "a cutting of a geometric figure into pieces that we can rearrange to form another figure" [6]. For example, a particular base shape, like a square, can be divided into individual component shapes in such a way that when the component shapes are rearranged, they create another base shape, like a triangle. This type of dissection is possible for any two polygons with the same area. It is a corollary of a remarkable theorem attributed to William Wallace, Farkas Bolyai, and Paul Gerwein, who each proved it independently in the early 19th century: *Any two simple polygons of equal area can be dissected into a finite number of congruent polygonal pieces*. Such polygons are called equidecomposable.

One of the most famous and elegant dissections of a square that can be reassembled into an equilateral triangle is due to Henry Dudeney, who published it as "The Haberdasher's Puzzle" in his 1907 book *The Canterbury Puzzles and Other Curious Problems* [2]. Figure 1 shows that dissection; throughout this paper it will be called the 4-3 dissection. The four components of the dissection are labeled A, B, C, D and colored green, blue, orange, yellow, respectively. The four components are all distinct, that is, no two of these components are congruent.

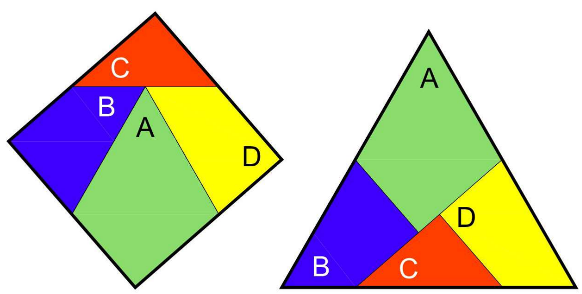
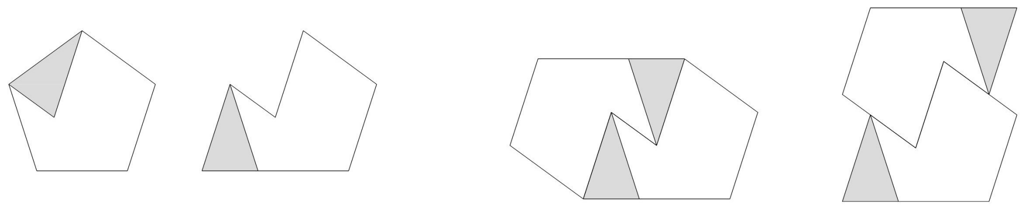


Figure 1: *4-3 square-to-triangle dissection.*

Grünbaum and Shepard note that the Wallace-Bolyai-Gerwein theorom immediately implies, "Every polygon P has the property that it can be cut into a finite number of pieces which can be rearranged to form the prototile of a monohedral tiling" [7, p.91]. Put another way, any shape – whether or not it can tessellate – can be divided in such a way that the reassembled pieces will tessellate. The tessellation by reassembled pieces is called a *dissection tiling*.

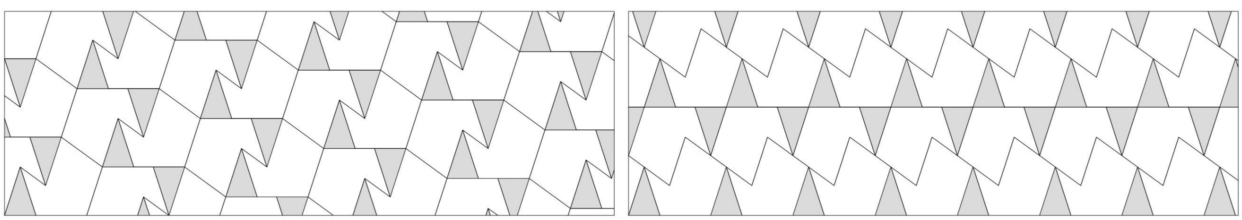
We give an example of this property using a regular pentagon. On its own, the regular pentagon cannot tessellate. However, when dissected into two pieces in a particular way [7, p.92], the two reassembled pieces form a shape that can tessellate in different ways. The dissection is as follows: using the top vertex as a pivot point, swing the upper left side of the pentagon inward to make an angle of 36° with the upper left side; join the endpoints of the pivoted side and original side to form a golden 36°–72°–72° triangle. Remove this triangle from the pentagon and attach it to the lower left side of the pentagon, as shown in Figure 2a.



(a) (b) (c)

Figure 2: *(a) Dissection and reassembly of a regular pentagon. (b) and (c) Two copies of the new shape can combine to form a tile that tiles the plane by translations.*

Figure 2b and 2c show ways in which two copies of the new figure can be assembled into shapes that easily tile by translations. Figure 3 shows tilings by these shapes.



(a) (b)

Figure 3: *(a) Tiling by shape in Figure 2(b). (b) Tiling by shape in Figure 2(c). The strip filled out by this shape can be translated or reflected to produce an infinite number of tilings.*

Dissection Tiling Systems

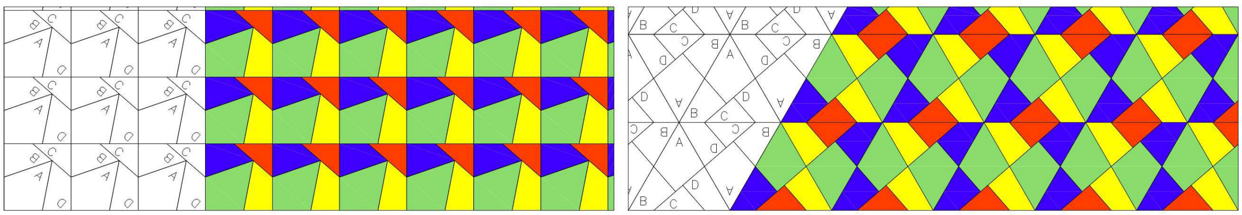
Ongoing research of mathematical dissections has revealed a variety of interesting findings, from the focus on special properties, such as swing hinges, twist hinges, or folding hinges [4,5], to the making of patterns on the surfaces of dissections [8], to the creation of an algorithm for creating dissection puzzles [10]. The purpose of this paper is to define and explore another type of special property for mathematical dissections, inspired by the combination of the geometric dissection and the dissection tiling. We call this special property a *dissection tiling system*, which consists of two equidecomposable base shapes that must adhere to the following three rules:

1. The base shapes must be able to tessellate.
2. The individual components into which the base shapes are dissected must be able to tessellate.
3. All combinations of individual components must be able to tessellate.

We focus on the Dudeney 4-3 square-to-triangle dissection in Figure 1 and show that these two equidecomposable base shapes form a dissection tiling system.

Base Shape Tessellations

The square and equilateral triangle are two of only three polygons (the other shape is a hexagon) that by themselves can tessellate. In their regular tessellations, congruent tiles meet edge-to-edge and all vertex figures are congruent. Figure 4 shows regular tessellations by squares and triangles, where the squares and triangles are the base shapes for the 4-3 dissection. On the left of each image, the line work and letter designations of the dissections are visible. On the right, the image transitions to show the assigned color coding of the components, graphically showing the patterning and distribution of the individual components.



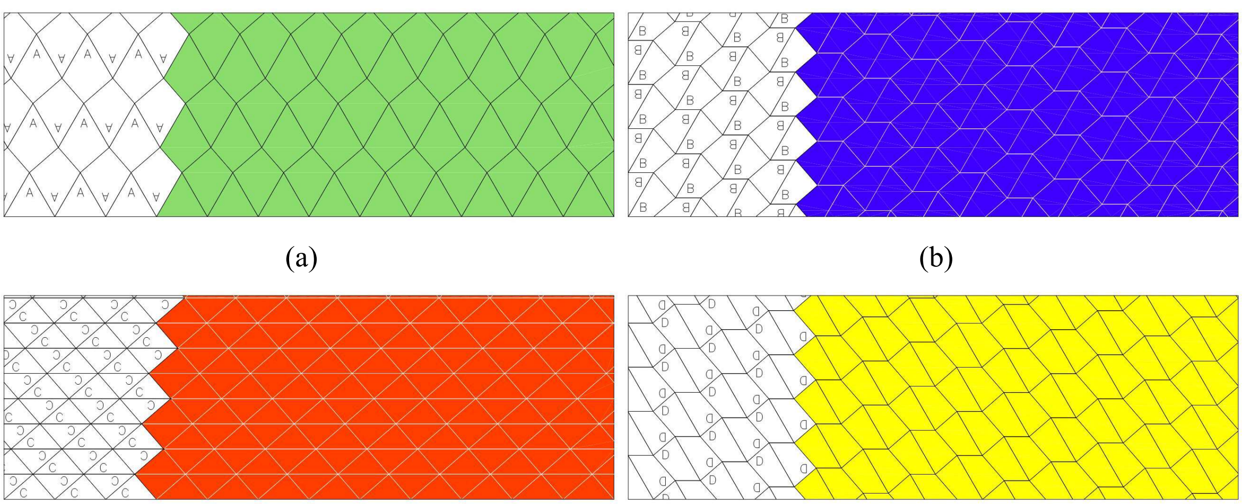
(a) (b)

Figure 4: *(a) Base shape square tessellation. (b) Base shape triangle tessellation.*

Individual Component Tessellations

The four individual components that make up the 4-3 dissection tiling system include one triangle, C, and three quadrilaterals, A, B, and D. It is known that any type of triangle will tessellate. To do so, one can take the triangle, rotate it 180° about the midpoint of one of its sides, and the resulting parallelogram will tessellate, as illustrated by the tessellation of the C component in Figure 5.

It is also known that every quadrilateral, convex or non-convex, will tessellate by repeatedly rotating it 180° about midpoints of its edges. The resulting group of four rotated quadrilaterals can be translated to produce a tessellation. Figure 5 illustrates the varying tessellations by the quadrilateral components A, B, and D.



(c) (d)

Figure 5: *Tessellations by the four components of the 4-3 dissection,*
(a) A-tile, (b) B-tile, (c) C-tile, (d) D-tile.

Combination Component Tessellations

When grouped together, the combinations of component tiles form a variety of larger shapes that include convex and non-convex polygons with different numbers of sides. Some combinations, like A+B, combine to create larger quadrilaterals, which can follow the rules of tessellating quadrilaterals as described previously. In other combinations, the possibility of tessellating was less clear and a method of trial and error was used to determine if the resultant combination of shapes would tessellate.

Figures 6 and 7 show tessellations by all ten possible combinations of the component tiles. Figure 6 shows tessellations by the six combinations of two tiles, including (A+B), (A+C), (A+D), (B+C), (B+D), and (C+D). Figure 7 shows tessellations by the four combinations of three tiles, including (A+B+C), (A+B+D), (A+C+D), and (B+C+D).

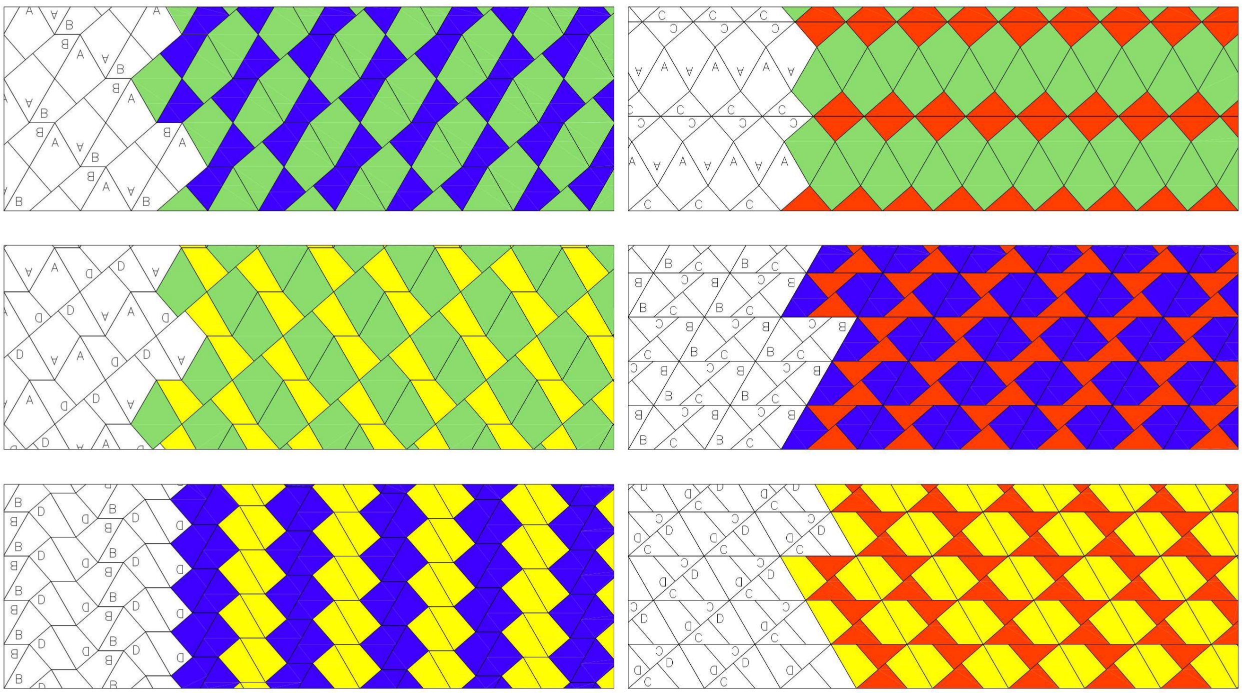


Figure 6: *Tessellations by the six combinations of two tiles.*

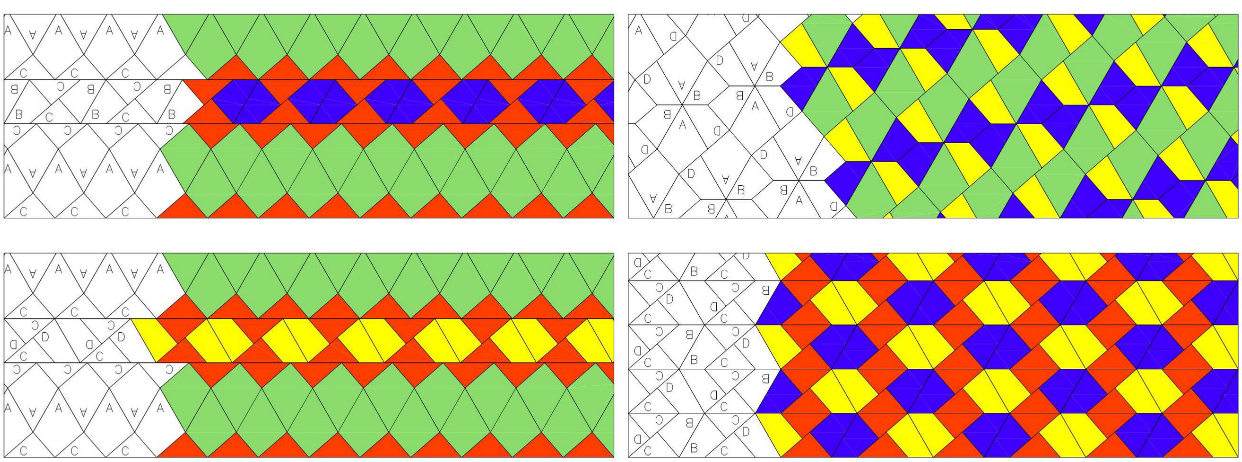


Figure 7: *Tessellations by the four combinations of three tiles.*

Symmetry Group Tessellations

For the 4-3 dissection tiling system, the division of each base shape triangle and square creates an asymmetrical pattern on each base shape. The patterned base shapes are then ideal prototiles, or primary cells, for the creation of patterns. In fact, the author has found that 14 of the 17 symmetry groups can be represented by the 4-3 dissection tiling system. Figure 8 shows one pattern example for each of those 14 symmetry groups, although more patterns are possible. It should be noted that to achieve some of the symmetry groups, mirror images of the component tiles are allowed. The standard notation for symmetry groups is included for reference. For further information on symmetry groups, standard notation, the additional patterns that are possible, and the construction of tessellations, see [1].

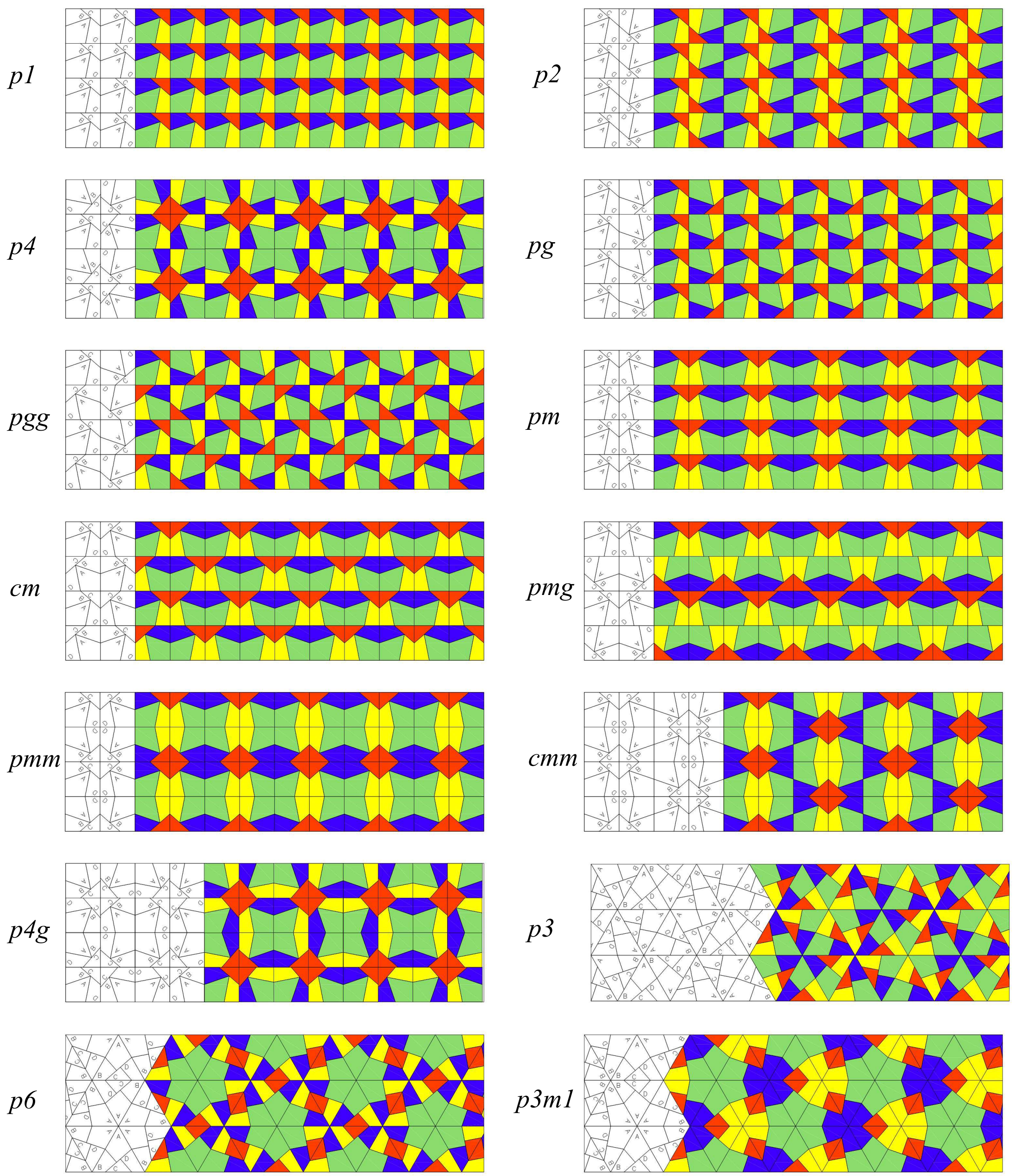


Figure 8: *14 of the 17 symmetry groups possible with the 4-3 dissection tiling system.*

The three symmetry groups not possible to illustrate with the dissected base square or triangle as a primary cell are p4m, p6m, and p31m. The first two would require that the pattern on the square or the triangle have mirror symmetry that is lacking. A p31m pattern is generated by reflections in the edges of an equilateral triangular cell with a pattern of 3-fold symmetry inside it, so our dissected base triangle cannot generate such a pattern. However, if we use nine copies of the dissected triangle to create a larger triangle that has a 3-fold symmetry, then a p31m pattern can be created by reflecting this larger triangle in its three sides. Figure 9 shows such a pattern, with the large triangular cell that generates the pattern outlined.

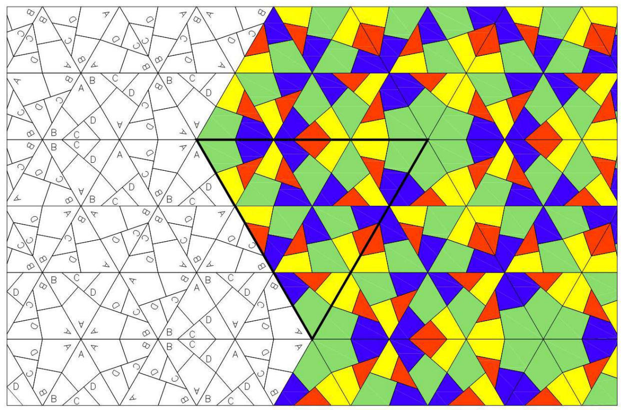


Figure 9: *A p31m pattern using nine copies of the dissected base triangle to form a large triangle as primary cell.*

Tessellation Patterns

In all of the previous examples, the color coding of each individual component was consistent, where A-tile was green, B-tile was blue, C-tile was orange, and D-tile was yellow. However, if this coding is eliminated or augmented, the opportunity for coloring and patterning is limitless. As with the line work of other tessellations, any of the line work shown in any of the previous examples of the 4-3 dissection tiling system could be used as the underlay for a color and pattern study. In Figure 10, some examples of the endless possible variations of these types of studies are illustrated. In these examples, two colors are used with the A-tile line work and two colors are used with the square tessellation line work.

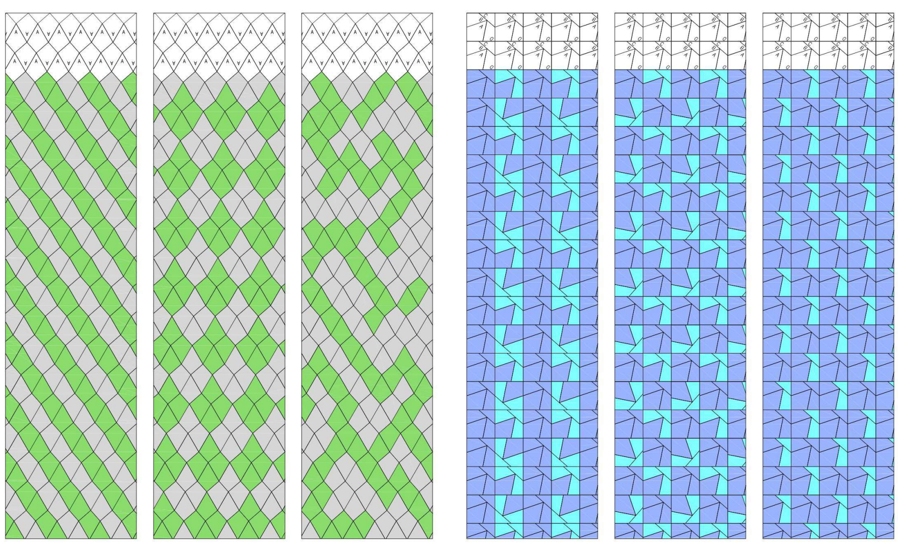


Figure 10: *(left) 2-color variations using A-tile line work. (right) 2-color variations using the square tessellation line work.*

Color coding can also highlight the different shapes that can be created with the 4-3 dissection tiling system, including a triangle, square, pentagon, and several sizes and shapes of a hexagon, as shown in Figure 11.

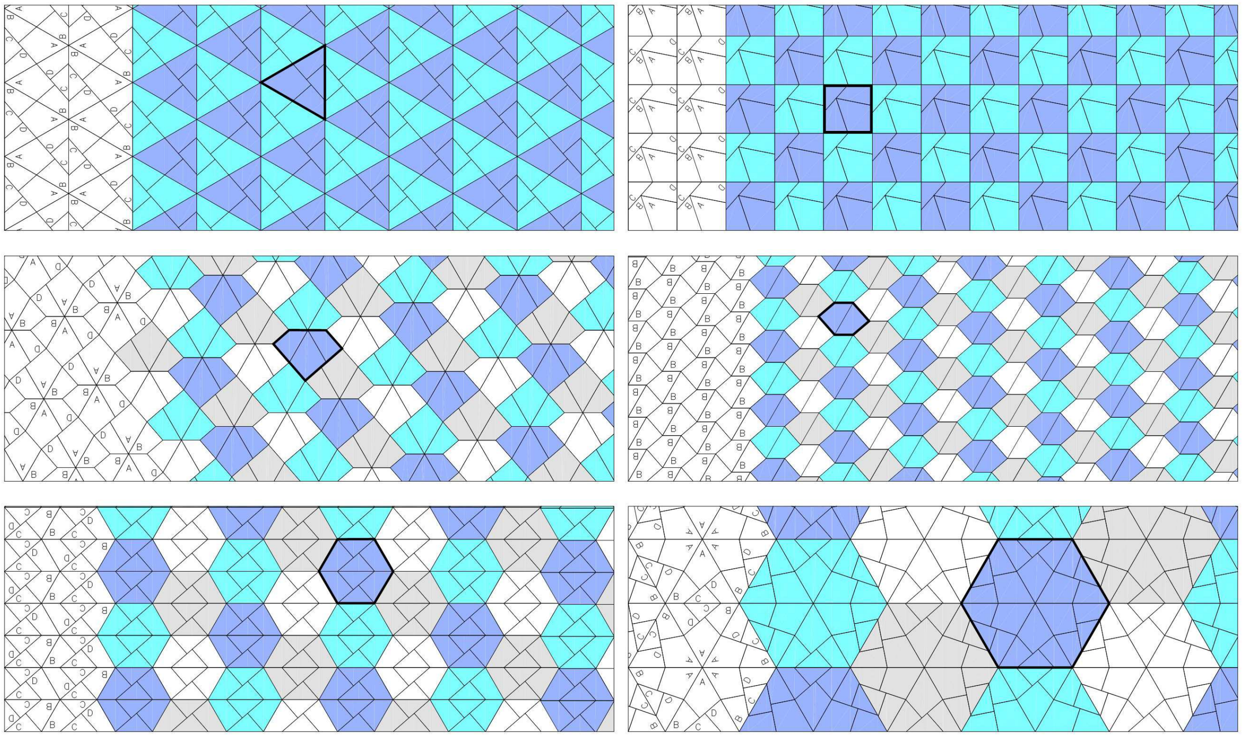


Figure 11: *Color coding to achieve different shapes.*

Other Possibilities

The Wolfram MathWorld dissection web page [9] and Frederickson's book [3] provide catalogues of many of the best-known dissections. There are only three dissections where the base shape that is a regular convex polygon will tessellate – thus satisfying Rule 1 – the 4-3 square-to-triangle dissection, the 6-3 hexagon-to-triangle dissection [3, p.142], and the 6-4 hexagon-to-square dissection [3, p.118]. This exploration started with the 4-3 square-to-triangle dissection, since it had the fewest number of pieces, and thus the fewest combinations to test.

There is potential for further study into dissection tiling systems. I have attempted a limited exploration of the 6-3 hexagon-to-triangle dissection tiling. While the base shapes of the hexagon and triangle tessellate, and the five individual component shapes tessellate, I was unable to get several combinations of shapes to tessellate. However, more work is needed to provide a conclusive result. I have not attempted a study of the 6-4 hexagon-to-square dissection tiling.

In addition to regular polygons, any geometric dissection *may* sponsor a dissection tiling system. Looking through various examples of geometric dissections, several stand out as potential candidates, including the one square to two smaller squares dissection [3, p.29], the square to rectangle dissection [3, p.32], and the Greek cross to square dissection [3, p.106].

The goal for a geometric dissection is typically to minimize the number of pieces. However, a dissection tiling system presents the opportunity for an alternate goal. Can one create a geometric dissection where the base shapes tessellate, the individual component shapes tessellate, and all combinations of individual components tessellate, regardless of the number of pieces?

Other questions arise based on the process used to determine compliance with the three rules of a dissection tiling system. Might it be possible to understand dissection tiling systems well enough to know how to generate sets that qualify? Is it possible to determine if a given geometric dissection will be a dissection tiling system without having to resort to the trial and error approach that was used?

I look forward to trying to answer these questions, and to exploring the tessellations and patterning potential of other geometric dissections.

Conclusion

In surface design, art, and architecture, the manifestation of a tessellation is a physical object with materiality, a fixed size and shape and thickness, and often with a color, pattern, or texture. For example, virtually every building surface – walls, floors, ceilings, and facades – are tessellations, made of small components grouped into larger compositions with no gaps and no overlaps. And, while many shapes are available, and an infinite number of shapes are possible, common shapes for products tend to be limited; think of ceramic tiles in a bathroom or carpet tiles in an office. Squares, triangles, and hexagons appear most frequently. Patterns, motifs, and texture can be applied to these products to expand the options available for designers and artists. However, a group of products based on a dissection tiling system suggest an alternate approach, one based on complimentary shapes that can be arranged and rearranged, using the principles of mathematics to expand the potential for artistic expression.

Acknowledgements

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