Art and Recreational Math Based on Kite-Tiling Rosettes

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Abstract

We describe here some properties of kite-tiling rosettes and the wide range of artistic and recreational mathematics uses of kite-tiling rosettes. A kite-tiling rosette is a tiling with a single kite-shaped prototile and a singular point about which the tiling has rotational symmetry. A tiling with n-fold symmetry is comprised of rings of n kites of the same size, with kite size increasing with distance away from the singular point. The prototile can be either convex or concave. Such tilings can be constructed over a wide range of kite shapes for all $n > 2$, and for concave kites for $n = 2$. A finite patch of adjacent rings of tiles can serve as a scaffolding for constructing knots and links, with strands lying along tile edges. Such patches serve as convenient templates for attractive graphic designs and Escheresque artworks. In addition, these patches can be used as grids for a variety of puzzles and games and are particularly well suited to pandiagonal magic squares. Three-dimensional structures created by giving the tiles thickness are also explored.

Introduction

Broadly defined, rosettes are designs or objects that are flowerlike or possess a single point of rotational symmetry. They often contain lines of mirror symmetry through the rotation center as well. Rhombus-based tiling rosettes with n-fold rotational symmetry of the type shown in Fig. 1a are relatively well known and can be constructed for any n greater than 2. Rhombus rosettes are edge-to-edge; i.e., the corners and edges of the tiles coincide with the vertices and edges of the tilings. This is also true of the rosettes described here, excepting the Escheresque variation of Figure 4.

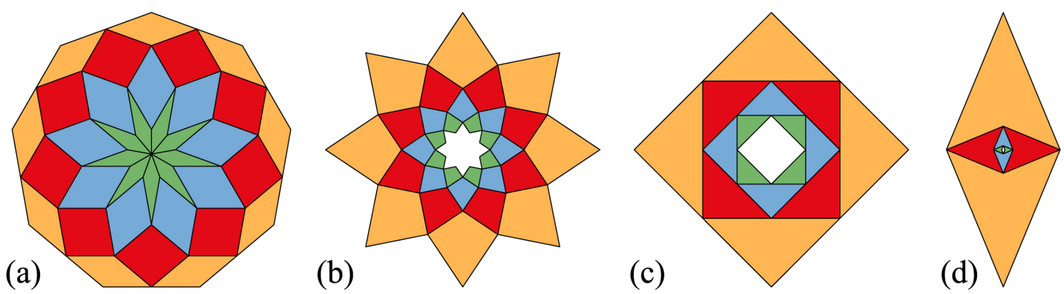

Figure 1: *Some tiling rosettes: a) an $n = 9$ rhombus rosette; b) an $n = 8$ kite rosette; c) an $n = 4$ isosceles triangle rosette; and d) an $n = 2$ dart rosette.*

In Grünbaum and Shephard's book *Tilings and Patterns* [6], a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. A patch of tiles is a finite collection of tiles that does not fully cover the plane. While patches of tiles are normally simply connected, the figures in this paper will be "annular patches" of tiles, with a hole in the middle. The tiles described here are "well behaved" by the criterion of Grünbaum and Sheppard; namely, each tile is a (closed) topological disk. However, the rosettes are not "well behaved" by the criteria of normal tilings; namely, they contain a singular point at the rotation center, defined as follows. Every circular disk, however small, centered at the singular point meets an infinite number of tiles; i.e., the tiles become infinitesimally small as such points are approached.

Previously, we described families of fractal tilings based on kite- and dart-shaped quadrilateral prototiles [1]. There are regions in these fractal tilings that can be described as rosettes of kites, darts, and equilateral triangles (Fig. 1). The kite rosettes in Fig. 1b, c, and d cover the infinite mathematical plane, while the rhombus rosette of Fig. 1a does not. While kite-tiling rosettes of the sort discussed here were certainly known previously, we're not aware of a paper devoted to their description and properties. Note that rings of kites are common in rosettes that are building blocks of Islamic tilings, these are invariably combined with other polygons. We describe here some properties of kite-tiling rosettes and the wide range of artistic and recreational mathematics uses of kite-tiling rosettes.

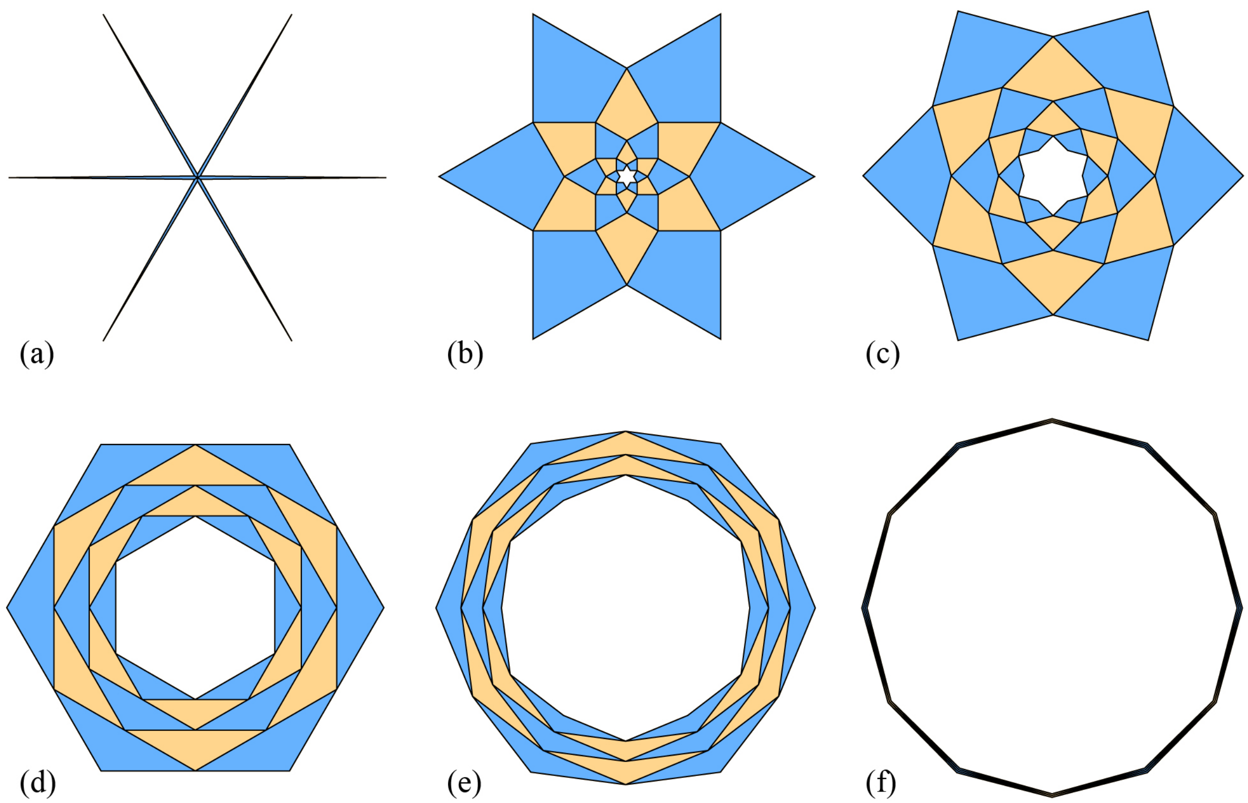

Figure 2: *Six-fold kite-tiling rosettes with kite side angles of a) 149°, b) 90°, c) 60°, d) 30°, e) 15°, and f) 1°.*

Properties of Kite-Tiling Rosettes

Kites are quadrilaterals with two pairs of equal-length sides that are adjacent to one another. They possess bilateral symmetry and can be either convex or concave. Concave kites are often called darts. The borderline case between convex and concave kites, where the two short sides are colinear, is an isosceles triangle.

In a kite-tiling rosette all tiles are similar; i.e., there is a single prototile. In an n-fold rosette, the tiles are arranged in rings of n kites, with a scaling factor between adjacent rings given by the ratio of the length of the short edge to the long edge of the prototile. The single prototile and n-fold symmetry together require the kites to be proportioned such that the side and apex angles sum to 180° – 180°/n. These rosettes possess a singular point at the rotation center, and they are two colorable. There is a continuum of kite shapes for all $n > 2$, and a continuum of concave kite shapes for $n = 2$.

Some examples are shown in Fig. 2 for $n = 6$, illustrating several special cases and limits. In Fig. 2a the side angles of the kite are 1° less than the limiting angle of 150° where the kite would be infinitely long. More generally, that angle is $180° - 180°/n$. In Fig. 2b the inner and outer boundaries of each ring of kites describe the first stellation of the regular n-gon, for which the kite side angles are $540°/n$. In Fig. 2d the boundaries are the regular n-gon, for which the kite side angles are $180°/n$. The side angles of the kites are 0° in the other limit, with an angle of 1° used for Fig. 2f. In this limit the boundaries describe a regular $2n$-gon.

A spiral is described by a chain of edges of the sort shown in Fig. 3. The angle between successive segments is fixed for a given n at $180° - 180°/n$. The ratio of successive segments in a spiral is given by the ratio of the two edge lengths in the prototile.

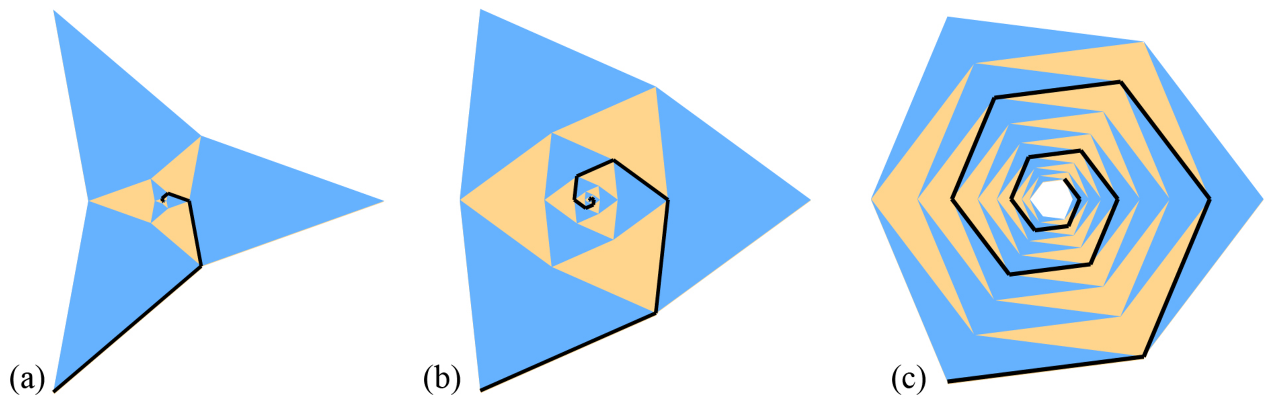

Figure 3: *Some spirals defined by n=3 kite-tiling rosettes.*

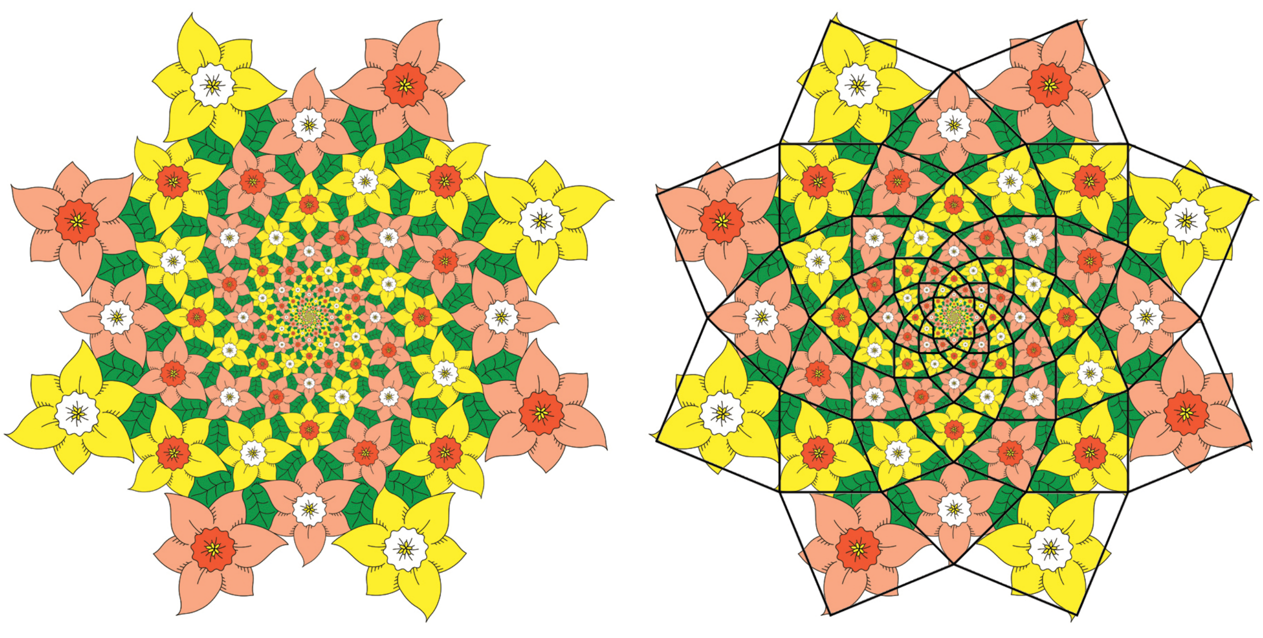

Figure 4: *An Escheresque design created by the author in 2011. The kite-rosette template is shown at right.*

Escheresque Art Based on Kite-Tiling Rosettes

As with any tiling, kite-tiling rosettes can serve as templates for Escheresque designs. An example is shown in Fig. 4, where each kite in an eight-fold rosette has been turned into one flower and two leaves.

M.C. Escher made several prints with rotational symmetry about a singular point at the center. It doesn't appear that he was thinking in terms of kite-tiling rosettes as templates, though. His three "Path of Life" prints [1] are based on smooth spirals elucidated in looping lines down the centers of the fishes and birds [2]. These can be fitted to kite-tiling rosettes; however, the angles of the kites are not round numbers, and there are two sets of markings in alternate kites. "Development II" (1939) is an unusual case, based on a hexagonal grid with 24-fold symmetry (if colors are ignored). In the full print, a central hexagonal grid develops into lizards as the design progresses outward. This grid can be obtained from a kite-tiling rosette by applying simple marks to an appropriate prototile as shown in Fig. 5.

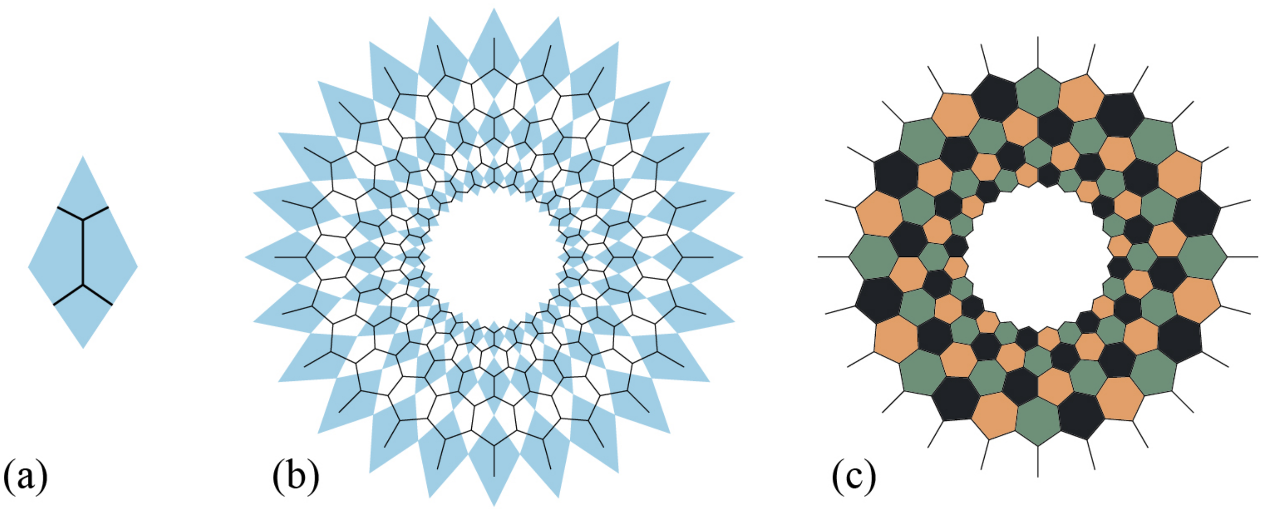

Figure 5: *a) An n = 24 kite with side angles of 120°, decorated with lines that generate a hexagonal grid (b) similar to that used in M.C. Escher's print "Development II". c) The coloring used by Escher.*

Knots and Links Based on Kite-Tiling Rosettes

Replacing the edges of the kites with interwoven strands allows the construction of a variety of knots and links. Adding rings of tiles can be seen as a way to create iterated knots and links, complementing techniques described earlier [4,5]. Figure 6 shows three different links with $n = 6$. For two rings of kite tiles there are three strands, for three rings two strands, and for five rings six strands. For four or six rings of kites there is a single strand.

If a prime n such as 7 is used, a unicursal knot is obtained for any number of rings up to six, at which point there are seven strands (Fig. 7). For seven to twelve rings it's again unicursal. For thirteen rings, however, there are again seven strands, but each strand now loops twice around the center compared to once for six rings.

These results can be understood as follows. Traversing a full revolution from any starting point takes $2n$ strand segments, where each segment corresponds to a single tile edge. Traversing from a corner (one of the outermost points) to the center (an innermost point) and back out to another corner takes $2(r + 1)$ segments, where r is the number of rings of kites. Dividing these, one corner-to-corner traversal covers $(1 + r)/n$ of a full revolution. Simplifying the numerical fraction, the denominator gives the number of traversals needed to close the strand. The total number of traversals n divided by that number gives the number of strands s; i.e., s is the greatest common divisor of $r + 1$ and n. For example, $r + 1 = 3$ in Fig. 6a, and 3 is the greatest common divisor of 3 and 6, so $s = 3$. Two more examples are given in Fig. 8.

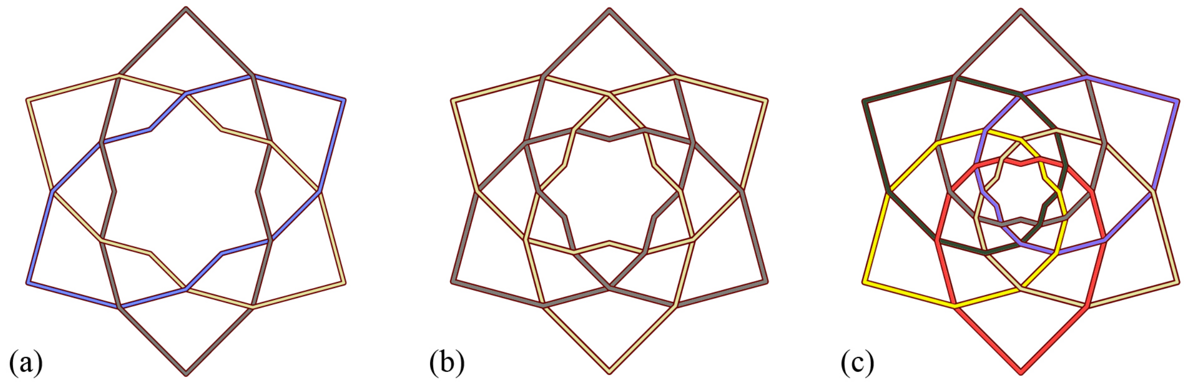

Figure 6: *Three links based on $n = 6$ kite-tiling rosettes, with a) three strands, b) two strands, and c) six strands.*

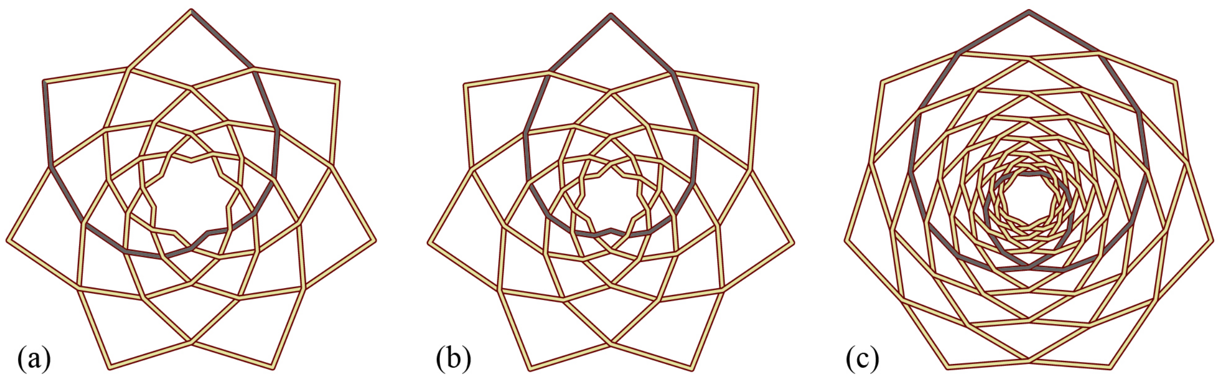

Figure 7: *A knot and two links based on $n = 7$ kite-tiling rosettes, with a) one strand, b) seven strands, and c) seven strands. In each, one traversal from the top corner into the center and back out to a corner is darkened to illustrate the path.*

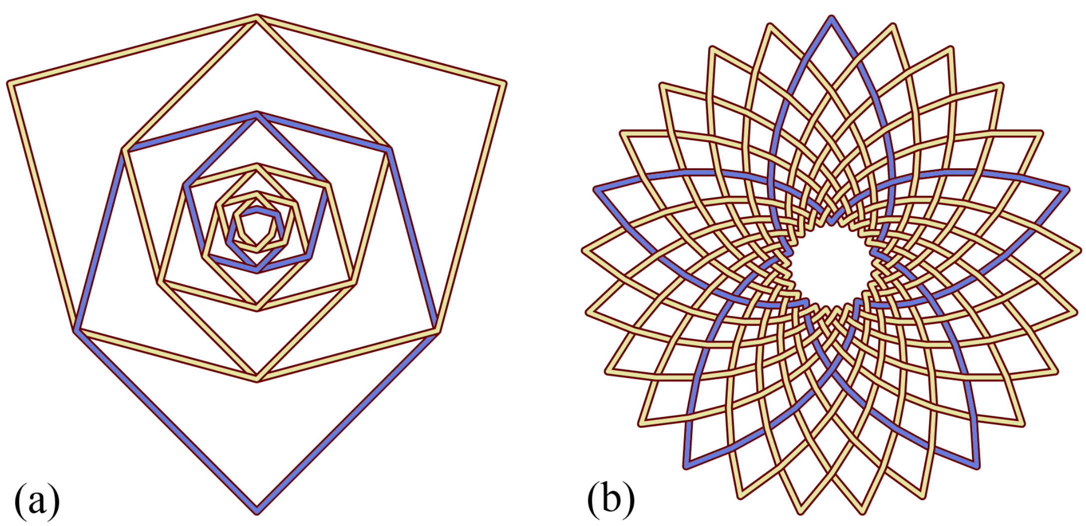

Figure 8: *Two links with a) $n = 3$, $r = 8$, and $s = 3$; b) $n = 25$, $r = 9$, and $s = 5$. A single strand is darkened in each link.*

Additional Recreational Mathematics Using Kite-tiling Rosettes

A kite-tiling rosette can be thought of as a grid of quadrilaterals, similar to a checkerboard. The key differences are the "serrated" edges and the fact that the grid wraps around to form a closed loop. There are many puzzles and games based on grids, including chess, crosswords, word searches, polyomino problems, and magic squares, any of which could in principle be adapted to kite-tiling rosettes.

After experimenting a bit with magic numbering of rosettes we found a direct mapping of magic squares to kite-tiling rosettes that has an advantage over a square grid. The strongest sort of magic squares are "pandiagonal", meaning the broken diagonals as well as the main diagonals sum to the magic number [7]. The numbers 11, 16, 6, and 1 form one broken diagonal of the square in Fig. 9a. The rows, columns, and two sets of diagonals of this 4 x 4 magic square can be mapped to an n = 4, r = 4 rosette as shown in Fig. 9b. One set of diagonals maps onto the four rings of kites, while the other maps onto the four diameters of the rosette. In contrast to the square, there is no preferred main diagonal, and all of the diagonals are easily read. Figure 9c helps explain why this mapping works. If additional copies of the magic square are placed adjacent to it the broken diagonals are completed. Within this diagram the group of 16 squares outlined in a heavy black line has the same layout as the rosette, except the two ends are not joined. Joining them as in Fig. 9d creates an object that is identical topologically with the rosette. One set of diagonals is given by the circumferential bands and the other by vertical loops. Note that the numbers of the indicated vertical loop read in order 1, 13, 4, 16, like the diameter of the rosette, but not like the main diagonal of the original magic square. This makes it clear the mapping will work for any n x n magic square.

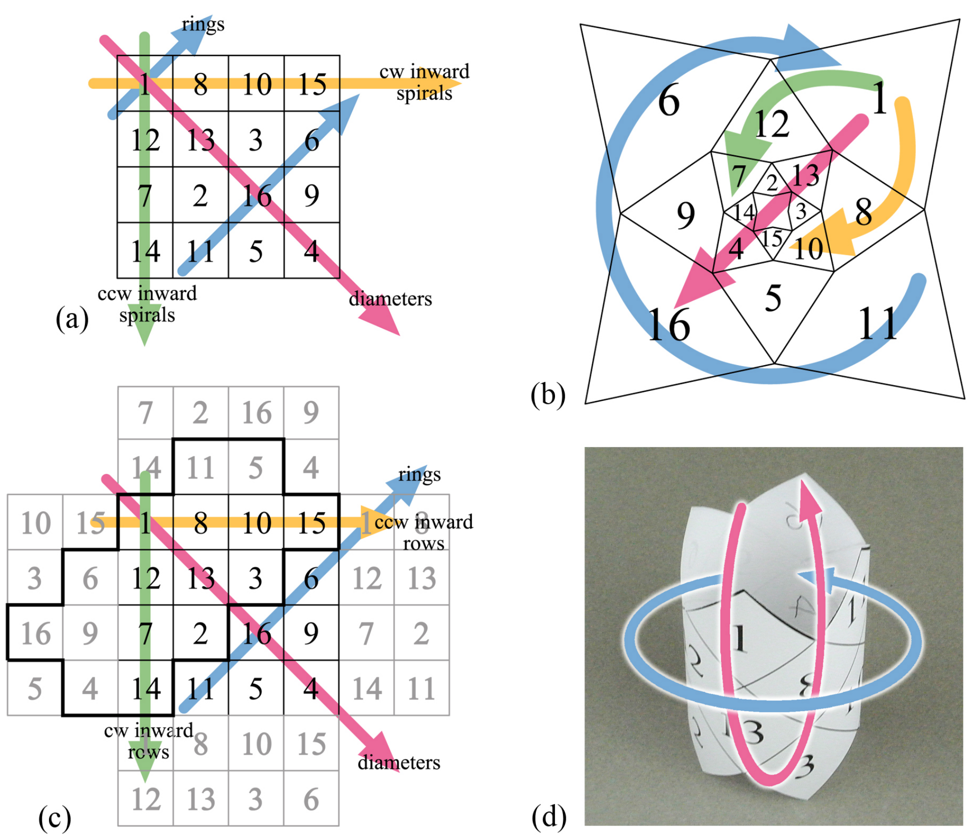

Figure 9: *a) A mapping algorithm for mapping pandiagonal magic squares onto kite-tiling rosettes. b) An n = 4, r = 4 magic rosette based on the magic square in (a). c) Extended magic square showing a group of 16 small squares that can be rolled in a cylinder (d) to create an object topologically identical to (b).*

Polyominoes are made up of squares connected in edge-to-edge fashion. Polyomino problems often involve tiling some figure with a specified set of polyominoes. Kite-tiling rosettes can be used for such problems, with the kites acting as squares. One class of polyomino problem is determining which polyominoes tile the plane with copies of themselves. The X pentomino is one such polyomino, as shown in Fig. 10a. This tiling can be successfully applied to a rosette if the rotational symmetry of the rosette n is a multiple of the repeat distance of the planar tiling. The repeat distance in this example is 5 in the direction relevant to tiling a kite rosette. Examples of n = 15 and n = 20 rosettes tiled with the X pentomino are shown in Figures 10b and 10c, respectively. These figures are esthetically pleasing in addition to their recreational math value.

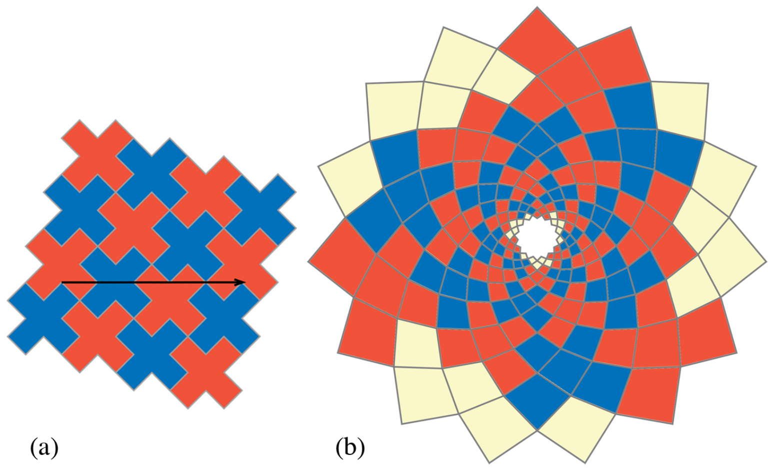

Figure 10: *a) A two-color planar tiling of X pentominoes. The black arrow indicates the repeat vector relevant to kite rosettes. b and c) Two-color X-pentomino tilings of kite-tiling rosettes with n = 15 and n = 20.*

Architectural and Sculptural Forms Based on Kite-Tiling Rosettes

The individual kite tiles can be thickened to create three-dimensional sculptures and architectural forms from rosettes. The simplest approach is to thicken each tile in a given ring by the same uniform amount, as shown in Fig. 11a. Simon Thomas exhibited a stainless steel sculpture at Bridges London in 2006 that appears to be of this sort. Adding a level of complexity, the thickness of uniformly-thickened kites can be varied within a ring, as shown in Fig. 11b. Non-uniform thickening of kites can also be performed, as in Figures 11c and 11d. In 11c kites are thickened in two different ways so that a pair of kites forms a building block.

Conclusions

This paper demonstrates the wide range of possible kite-tiling rosettes and describes some of the many artistic and recreational uses of them. There are several possible avenues for further exploration, including additional Eschesque tessellations based on other rosette parameters and different types of puzzles and games. Tiling and knot rosettes could also be incorporated in larger or repeating designs in the Islamic or other styles. Three dimensional structures could be further developed and 3D printed.

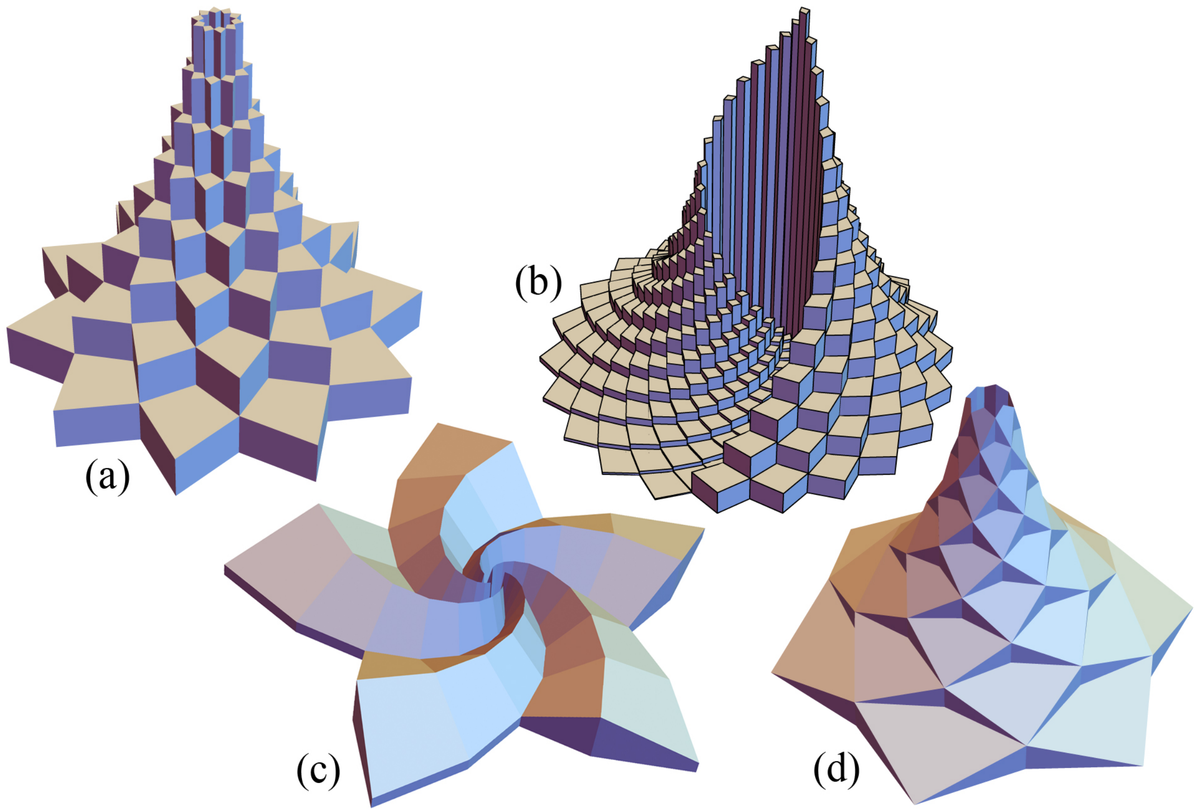

Figure 11: *Three-dimensional structures created by a) thickening each kite uniformly within each ring of a rosette; b) thickening individual kites uniformly, but varying the thickness within a ring; c) thickening kites non-uniformly in a manner that halves the rotational symmetry, and d) thickening kites non-uniformly to create a structure similar to that in 11a.*

References

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