Two-Layer Woven Surfaces with Planar Faces

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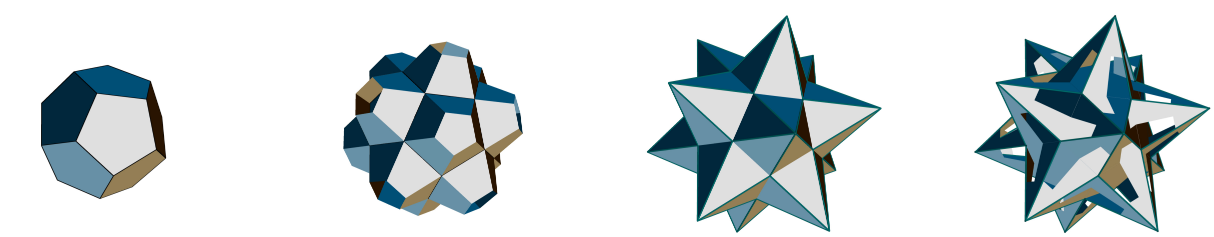
Abstract

We create two-layer interwoven surfaces with (a) planar faces from (b) arbitrary input meshes such that these can be built from cardboard or any other planar material. There are pre-existing constructions for symmetric and regular meshes but these are lacking one or both of the before mentioned attributes (a), (b). We want to emphasize that there do not exist solutions for edge-interpolating or barycentric-apex-created two-layer weavings in general. Hence, we propose constructions in terms of approximations, missing one of the conditions (a), (b) alone, but not both.

Introduction

The Task

Inspired by the work of Rinus Roelofs [1], we look for constructions to create similar interwoven surfaces with planar faces for input meshes which may lack symmetry or regularity of their faces. The constructed surfaces may or may not interpolate edges of their input meshes and select apexes not necessarily above the center of the input mesh faces.

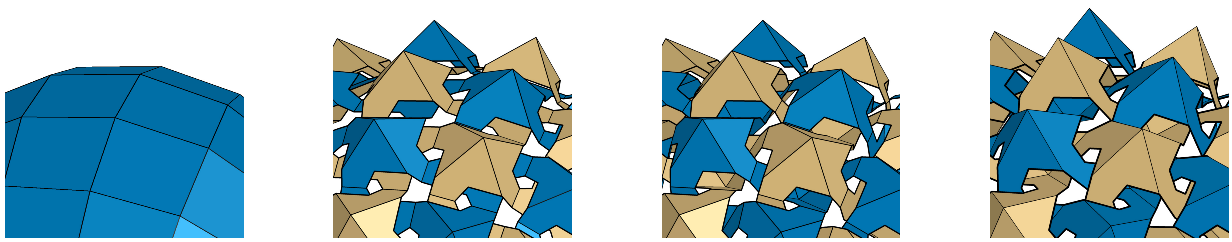


(a) *Dodecahedron.* **(b)** *Extend faces.* **(c)** *Kepler Star.* **(d)** *Cut half edges.*

Figure 1: *Construction of the star in M. C. Escher's "Gravity".*

Related Work

The work of Rinus Roelofs [1] is inspired by Luca Pacioli, Leonardo da Vinci, and Johannes Kepler: by the processes of elevation and stellation from an input polyhedron a two-layer surface is created. These two-layer surfaces have edges of self-intersection. Another inspiration comes from Maurits Cornelis Escher who created images showing models of these two-layer self-intersecting surfaces with the intersection areas cut away (Figure 1d): each intersection edge is split at the center, then always an area around one half edge is cut away in one layer and an area around the other half is cut away in the other layer. When walking cyclically along the edges of a face in counter-clockwise direction, we always cut away the first half edge on the lower and the second half edge on the upper layer. This leads to surfaces which do not have self-intersections anymore, but holes with linked boundary curves. For vertices with even valence, this creates two linked hole boundaries, whereas an odd valence yields one knotted hole boundary. Boundaries which are generated from neighboring vertices are linked. If the neighborhood graph – which consists of vertices (representing faces of the mesh) and edges connecting two vertices, when the corresponding faces share an edge in the mesh – of the input surface contains any odd cycle, the resulting weaving is connected, but if all cycles in the graph are even, we get two disconnected interwoven surfaces. The process of elevation is used by Rinus Roelofs in a generalized way, where the created triangle faces are no longer necessarily equilateral; this adds degrees of freedom for the positions of the pyramid apex points. The construction can also be used for planar tilings, and generalized in terms of edge elevation. There are also some models where instead of the pyramid a bi-pyramid is used, with an upper and a lower apex. If these apex positions are well chosen, a pair of connected triangles which share one of the original tiling edges can be joined to one planar quadrilateral. This is the construction which we want to extend to more general cases, allowing input meshes with irregular and non-planar polygons.



(a) *Underlying mesh.* **(b)** *Single elevation.* **(c)** *Double elevation.* **(d)** *Planar union of triangles.*

Figure 2: *Single, double elevation and our planar construction.*

Constructions

Overview

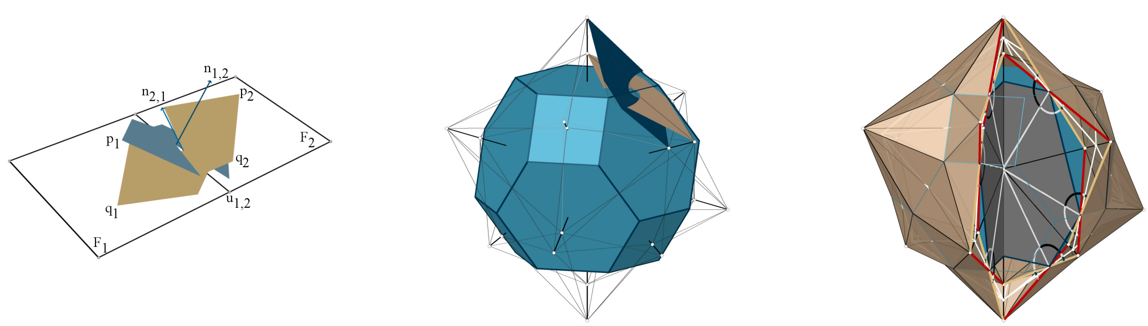
There are several approaches for the construction of interwoven surfaces, despite the fact that a layered structure was possibly not their purpose. One is the method of *stellation* defined by Johannes Kepler in 1619 for polygons and polyhedra where we extend edges and faces until they meet, forming new polygons and polyhedra. Applying this procedure for instance to a dodecahedron yields the small stellated dodecahedron, also called *Kepler Star*, cf. Figure 1c. The artist Maurits Cornelis Escher created a print of the solid with the name *Gravity*, see Figure 1d for an abstraction. There we can see the idea of a two-layer structure, giving us an insight of the basis of the pyramid placed upon every face of the dodecahedron. This kind of reception is another method called elevation (Figure 2b). Originally inspired by the work of Luca Pacioli and Leonardo da Vinci in their work *La Divina Proportione* in 1509 where pyramids consisting of equilateral triangles were placed upon every face, elevation generalizes this concept and allows different pyramid heights. Attached to a Platonic or Archimedean solid, with an apex sitting on the outside above the barycenter of each face, it builds new triangular faces with the solid's edges. The small stellated dodecahedron can be deduced by this construction and hence the underlying solid is still there, unseen, but with Escher's illustration giving us a glimpse. Motivated by this idea, the artist Rinus Roelofs adopted the process of elevation and extended it to flat tiling patterns with regular faces in [3] where he also mentions the possibility of double elevation (Figure 2c) to create a two-layer structure where we simply place two pyramids upon one face in both directions. Here, it is important to note that the original faces do not count anymore and we glue faces of the pyramids in a face-alternating order. Another extension is the edge-elevation, described in [2], where Rinus Roelofs uses a new transformation defined by connecting the midpoint of each edge with its adjacent faces and extrude these midpoints outwards giving a pyramid with four triangles upon each edge. This extension rose from Rinus' observation that it is equal to the result of a first and second elevation of the octahedron, see [2]. His work about interwoven surfaces [1] can be examined as smooth versions of single or even double elevated flat tiling patterns.

To construct a two-layer weaving, we place two points, one below and another above each face of a mesh with regular faces, and connect them to the corresponding face edges. Observe that the heights of the apexes can be zero, but not for upper and lower at the same time where we get in the lower apex case the single elevation (Figure 2b). Furthermore, a new face of the weaving is gained by connecting an upper apex of one face with an incident edge and the lower apex below the neighboring face. They are then either planar quadrilaterals (Figure 2d) or result in two triangles forming a crease in the edge. In terms of regular polyhedra and tilings there are no examples of weavings with creases in the edges by Rinus Roelofs, as he chooses the same apex heights above and below each face, carefully adjusted to the face types. To guarantee the property of a weaving without self intersections, we take every edge and the new two faces where the edge lies in and cut in holes such that both faces pass each other without intersection, like in Figure 1d. The aforementioned constructions by Kepler, Pacioli, and Roelofs share the concept of underlying meshes with regular faces, and apexes declaration in face normal direction above or below the face barycenter, with its new faces being planar or showing a crease in the edges of the underlying meshes.

Relaxation of Regularity

Given an edge e with two incident faces F_i and pyramid apexes p_i, q_i in normal direction above and/or below the center of these faces for $i = 1, 2$, respectively, see Figure 3a. When we relax the condition of regularity, allowing different heights for the two apexes over every face and meshes with any kind of polygonal faces, we ask for the existence of two-layer weavings with their faces being planar, quadrilateral, and passing the edges of the mesh with the two new sheets per edge e given by $p_1 e q_2$ and $p_2 e q_1$. These sheets receive cuts, so that they pass each other without self-intersections. They might also get a cut at both sides to avoid intersections with sheets of neighboring faces.

A first valid candidate of meshes are planar triangulations, with the idea in mind that three pairwise non-parallel planes intersect in a point. Consider a line per face given by the face normal direction and the intersection point of angle bisectors in that face. Then one selected apex above a triangle will identify the heights below the neighboring faces because the planes given by the apex and the face edges intersect these lines in unique lower apex points. Continuation of this process gives us at least one height over each triangle. In the other case, we start with two heights, one above and the other below the triangle, producing a weaving consisting of two connected components. One selected height is enough if we have one vertex of odd valence or an odd cycle in the dual graph of the mesh. In the case of triangulations taking the barycenter does not work in general but apexes above or below the intersection point of triangle angle bisectors, which are the centers of a triangle's inner circle, will suffice.



(a) *Sheet construction.* **(b)** *Two sheets over an edge.* **(c)** *Outer shell and insight.*

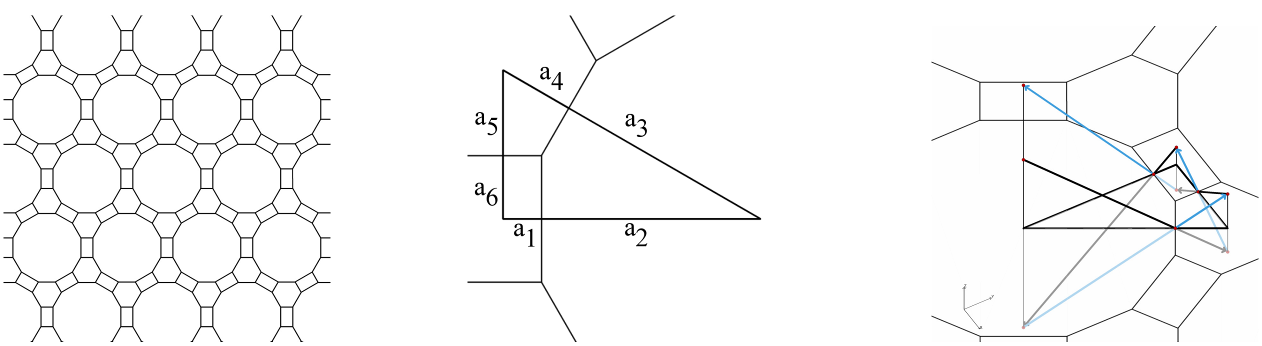
Figure 3: *General construction, truncated octahedron with two sheets and construction skeleton.*

Another candidate are the Archimedean solids, as we can inscribe them inside a unit ball with their vertices lying on the boundary of that ball where all face normals attached to the face barycenters meet in the origin. The latter property together with the faces being regular gives the following construction for two-layer weavings. Consider the plane perpendicular to the line connecting an edge midpoint and the center of the solid, cf. white lines in Figure 3c. Now, depending on the existence of only even or at least one odd cycle in the neighborhood graph of the solid, we are allowed to choose two or one angle(s), respectively. Attaching this chosen angle(s) to all edge-midpoints above and below the before mentioned perpendicular planes, we deduce two new planes per edge, both constructed by the edge itself and the opening angles (Figure 3b). Applied to all edges and due to the face regularity, all new planes meet in two points per face, located in face normal direction above or below the face barycenter. Figure 3c shows the sheet construction with a cut where red and yellow lines (connections between apexes) are alternately appended forming two cycles while traversing the neighborhood graph along the cut.

A third valid type of meshes are flat tilings with regular faces. For these, we want to introduce the *closing condition*. Given a mesh M embedded in $\mathbb{R}^2$. For each vertex $v \in M$ take the incident faces with their respective barycenters, connect a center with the edge to the neighboring face and afterwards with its center. Continuation counterclockwise around the vertex yields a closed path and we denote the Euclidean distances to the edges with $a_1, a_2, a_3, a_4, \ldots$ like in Figure 4b. The goal is to get two points for each face, one denoted as the lower and the other as the upper point. We want to start our walk with an upper point and connect it to the lower point of the neighboring face. This then gets connected to the upper point of the next face in counterclockwise order, so that we finish our walk in v with odd valence giving one circulation, otherwise two. Given the height h_1 for the upper point, then the height h_2 over the neighboring face is given by $h_2 = -\frac{a_2}{a_1} h_1$. A concatenation yields

$$1 = \left((-1)^{d(v)} \frac{a_{2d(v)}}{a_{2d(v)-1}} \cdots \frac{a_4}{a_3} \frac{a_2}{a_1} \right)^2 \tag{1}$$

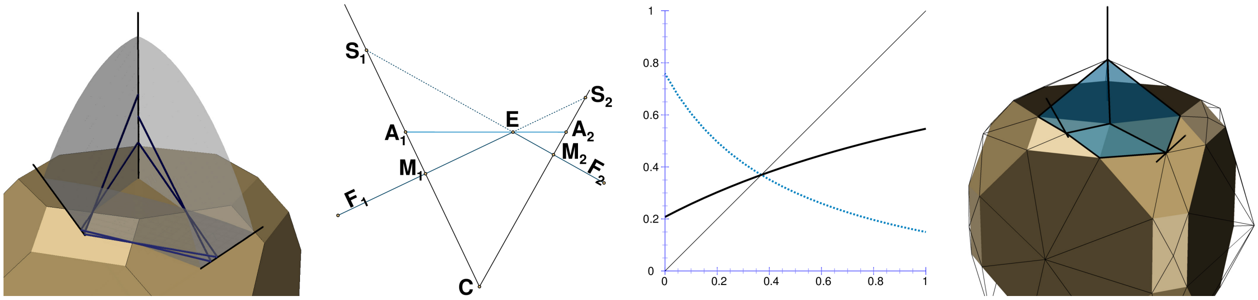
where $d(v)$ denoting the valence of v. If all faces are regular, we have $a_{2i} = a_{2i+1}$ and $a_{2d(v)+1} = a_1$. Therefore for a tiling consisting of regular faces only, Equation 1 is always fulfilled. Consequently, we deduce that flat tilings with regular faces are indeed a valid choice for two-layer weavings. But can we relax the condition



(a) *Flat tiling.* **(b)** *Flat distances.* **(c)** *Non-closing path.*

Figure 4: *Non-regular tiling with 12-, 6-, and 4-sided faces.*

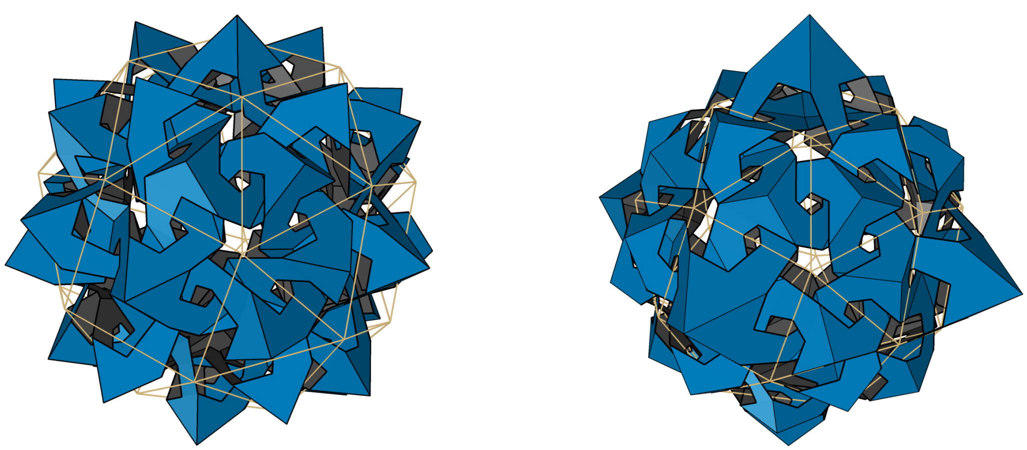
of regularity? Consider the example of the flat tiling given by 12-, 6-, and 4-sided polygons as illustrated in Figure 4a, where the quadrilaterals and hexagons are not regular. Starting our walk in the chosen upper vertex in the 12-sided polygon, see Figure 4c, with black arrows indicating a top-to-bottom direction and blue a bottom-to-top, we finally end up above where we started. The only valid starting height in this non-regular 12-6-4 tiling is zero, due to the closing condition, but then the two layers coincide.



(a) *Non-closing path.* **(b)** *Height calculation.* **(c)** *Height analysis.* **(d)** *Collapsing faces.*

Figure 5: *The modified great rhombicuboctahedron during the two-layer weaving process.*

A second counterexample deals with the slight modification of the *great rhombicuboctahedron*. Similar to the previous flat example, we shrink the quadrilateral such that all of them and the hexagons lack regularity, see Figure 5a. Starting our path in the upper apex above the 8-gon, passing alternately lower and upper apexes, we want to finish our walk in the apex we started in which is unfortunately not the case (Figure 5a). In comparison to the closing condition for flat tilings, we might express the height of the apex in the neighboring face with the help of the cross ratio, see Figure 5b, where the $F_i's$ denote the faces and M_i their midpoints, for $i = 1, 2$. The value A_1 is the apex with known height, S_i the intersections of the extended neighbor faces F_i with the lines given by face normals through M_i, all meeting in C, for $i = 1, 2$, and E denotes the edge between the faces pointing away. The height h_2 for A_2 can be derived from $0 = H_1H_2 - h_1H_2 - h_2H_1 - h_1h_2tan^2(\alpha)$, with $H_i = ||S_i - M_i||_2$, $h_i = ||A_i - M_i||_2$ for $i = 1, 2$ and $\alpha = \angle(M_1CM_2)$. A concatenation is unfortunately not as beautiful as Equation 1, so that we give for this example a height analysis, shown in Figure 5c. The dashed blue and black lines indicate one and two circulations, respectively, illustrating at which height we arrive after an initial choice, indicated by the argument axis. The height values are absolute and taken over a face in face normal direction. The thin line of bisection shows the preferred height we want to reach after two circulations. In our example there exists one height at which we arrive after even and odd numbers of circulations, at the intersection of all three lines. Hence, there exists one height for this modified example, but then all faces of the two layers coincide, like in the flat tiling case (Figure 5d).



(a) *Not edge-interpolating.* **(b)** *Edge-interpolating.*

Figure 6: *Two-layer weavings on the snub cube.*

Approximation of Planar Face Weavings

How can we extend the two-layer construction with planar faces to meshes with arbitrary polygonal faces and which conditions might be loosened? Here, we want to propose two approaches using approximations, both being displayed in Figure 6. In the first one, we do not ask for edge interpolation, i.e. the new faces connecting the four apexes of neighboring faces do not have to intersect in the edge anymore, but keeping the intersection line parallel to the edge. Since this gets applied to the whole mesh neighboring intersection edges of a face might be skew and eventually do not meet in a point. We can overcome this with cuts in the quadrilateral sheets at the positions of vertices. Figure 3a gives the idea of how to glue neighboring sheets together because of this phenomenon. Lastly, we have the freedom to choose apexes for every face individually in terms of height. Note that with the utilization of cuts, two sheets related to an edge depend only on the four apexes and direction of the edge connecting the faces. Hence this approximation does not improve in quality over several iterations as we do not update apex positions or edge directions – a matter covered in the following approach.

The second approximation maintains the edge interpolation but relaxes the condition that the apexes not necessarily lie above or beyond the barycenter in face normal direction. The idea is to ask all apexes how we have to change their positions and heights such that all new two-layer faces interpolate the mesh edges. This goal can be achieved with gradient descent. Therefore we solve the following minimization problem

$$min\left(\sum_{p_i}\sum_{k=1}^{e(f_k)}||n_{i,k}p_i - d_{i,k}||_2^2 + \sum_{q_i}\sum_{k=1}^{e(f_k)}||\tilde{n}_{i,k}q_i - \tilde{d}_{i,k}||_2^2\right),$$

with p_i and q_i ranging over all upper and lower apexes, respectively, k addressing all neighboring faces of the face f_i, to which p_i and q_i belong to, and $e(\cdot)$ counting the incident edges of that face. The normals $n_{i,k} = \frac{(u_{i,k}-q_k)\times e_{i,k}}{||(u_{i,k}-q_k)\times e_{i,k}||_2}$ and $\tilde{n}_{i,k} = \frac{(u_{i,k}-p_k)\times e_{i,k}}{||(u_{i,k}-p_k)\times e_{i,k}||_2}$ together with $d_{i,k}$ and $\tilde{d}_{i,k}$ define the planes in Hessian form where one is given by the neighboring faces' apex in opposite height together with the edge $e_{i,k}$. The value $u_{i,k}$ is a vertex of this edge (Figure 3a). Hence, we modify the apex positions such that all of them lie in the plane spanned by their respective opposite edge-neighboring apexes and the face connecting edges. Since we modify only for apex positions we iterate several times starting with the initial sets P^0 and Q^0 of upper and lower apexes, respectively. The normal creation uses the sets of apexes of the previous iteration step to modify the values.

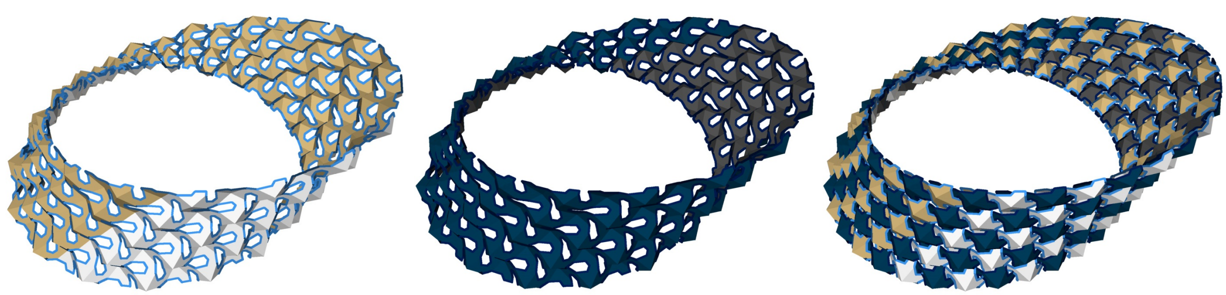


Figure 7: *Components and weaving of the odd-cycled Moebius strip.*

Results

With two possible variations of weaving-constructions at hand, the following examples focus on maintaining edge-interpolation. In general it is up to the viewers' perception which of the two approaches is more suitable. Figure 6 perfectly shows that the version, which does not interpolate edges yields more consistent pyramids, whereas the interpolating version evens the cuts in the sheets but increases the quadrilateral pyramids in height at the same time. Figure 8 shows weavings deduced from flat tilings and meshes of closed surfaces. The result in each column in the top and bottom row originates from the two meshes in the second row. In the flat case we use tilings composed out of (non-)convex (star) polygons, the above mentioned counter example of a distorted Archimedean tiling and a Penrose pentagon tiling, all shown in the first row in Figure 8.

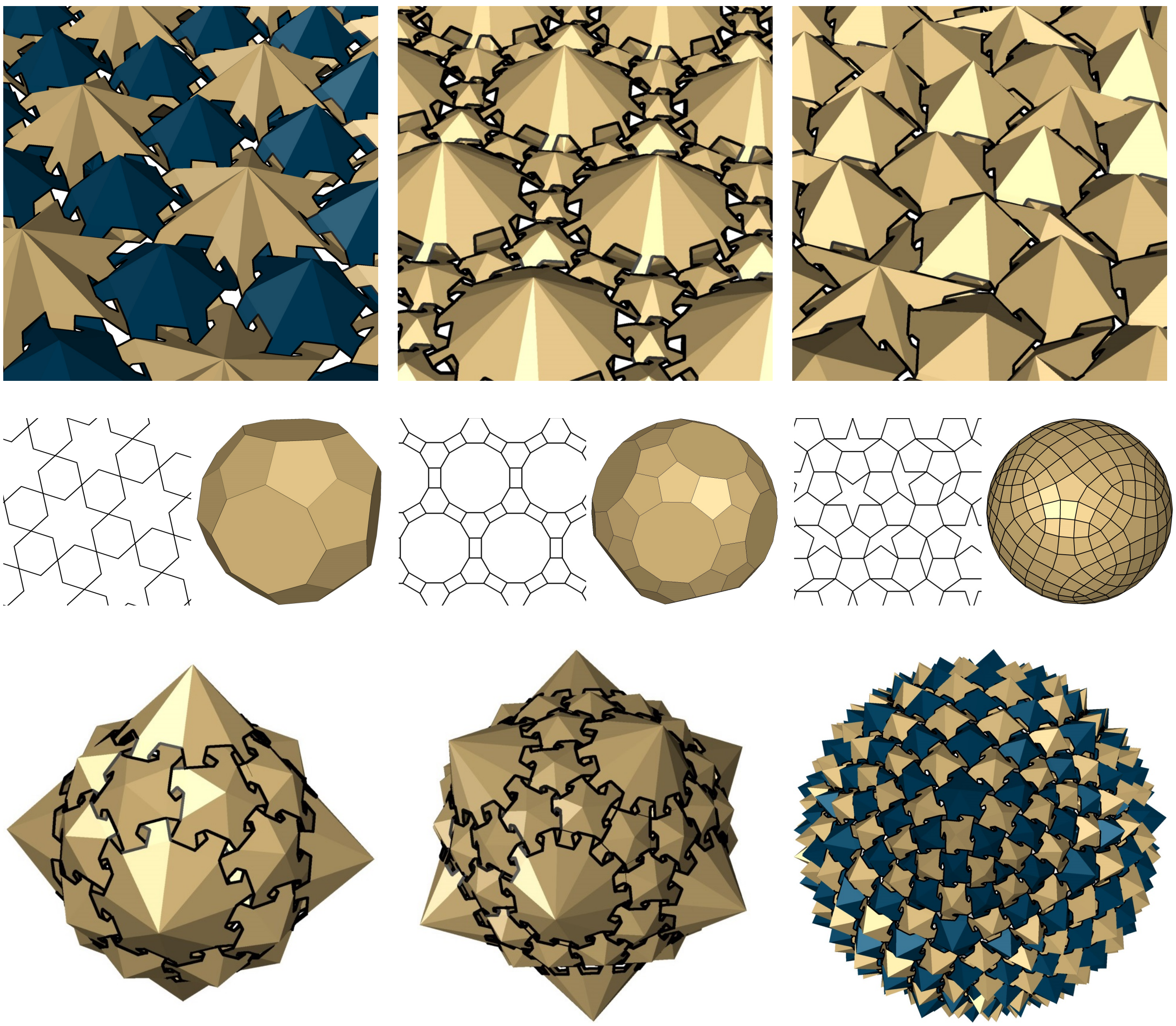


Figure 8: *Two-layer weavings on flat tilings (first row) and closed surfaces (third row). The second row displays the underlying meshes from left to right: (star) hexagon tiling, near-miss Johnson solid 5-5-8, distorted Archimedean tiling, near-miss Johnson solid 5-5-10, Penrose pentagon tiling, refined snub cube.*

Examples of closed surfaces are the two near-miss Johnson solids which are convex polyhedra whose faces are close to being regular polygons, the first consisting of pentagons/octagons, and the second of pentagons/decagons, and a refinement of the snub cube with their respective weavings displayed in the third row in Figure 8. Neglecting orientation or closeness we construct examples of the Moebius strip in Figure 7, with an odd cycled version, as well as the helicoid and the Klein bottle illustrated in Figure 9. The former reverses the separation in components with respect of cycle-length in the neighborhood graph. An even cycle on the Moebius strip consists of one component due to the flip of orientation resulting in an orientable strip with two circulations. An odd cycle with flip of orientation however produces two interweaving components.

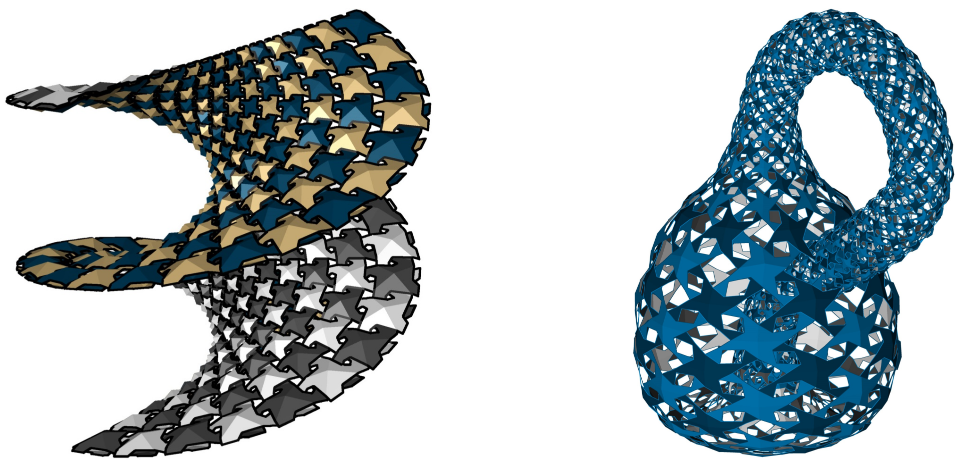


Figure 9: *Two two-layer weavings on the helicoid and Klein bottle.*

Future Work

All the presented constructions are suitable for two-layer weavings. A natural question is how can we extend the construction to multiple-layer weavings with more than two layers. Beside the known restrictive obstacles, like regularity of faces or edge-interpolation, another design parameter emerges, namely the way we connect multiple layers across an edge. We relaxed the condition of regularity, gave (counter)examples, and approximations. But we would like to investigate further now to find conditions guaranteeing the existence of two-layer weavings for meshes with arbitrary faces, like the one in Equation 1.

Acknowledgments

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