3D Printable Golden Sponges

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Abstract

I consider iterated function systems (IFSs) in $\mathbb{R}^3$ with a common contraction rate λ. For certain values of λ for which the iterated function system has sufficient overlap, an approximation to the resulting attractor is structurally capable of being 3D printed.

Introduction

The Sierpiński Triangle, as shown by Kenneth Falconer in [1] and illustrated in Figure 1, is one of the most well known and reproduced 2-dimensional fractals in mathematics. Its 3-dimensional counterpart, the Sierpiński Tetrahedron, was first illustrated by Dr. Alan Norton and included by Mandelbrot as plate 143 in [2]. Since these fractals contain infinite detail at all levels, we may only model them by forming a finite approximation to the actual objects. Even with this restriction, producing successful 3D prints of these fractal objects presents some structural challenges. These challenges arise due to the fact that individual tetrahedra in the model are only connected to each other at a single point, thus making the model unstable when printed. In this paper I show a technique that preserves a property of the fractal object called total self-similarity and also allows the model to be strong enough to be 3D printed.

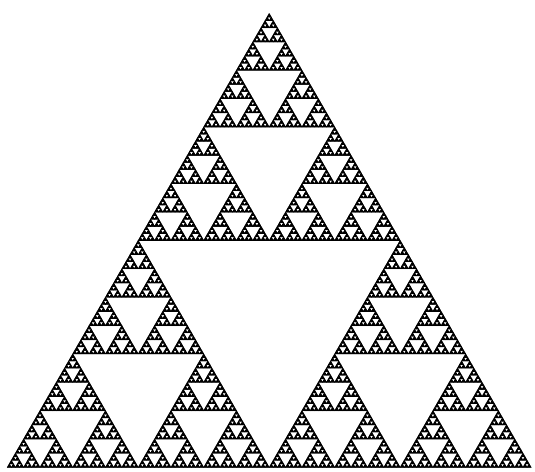

Figure 1: Standard Sierpiński Triangle

Each of the fractals we consider can be realized as the attractor of an iterated function system. To produce models of fractal objects that are 3D printable the contraction rate is varied so that the individual components in the object have sufficient overlap.

The Sierpiński Tetrahedron can be realized as the attractor of the following IFS:

For $0 < \lambda < 1$, we consider the compact invariant set "attractor" Λ_λ of the iterated function system

$$F_\lambda(x) := \{f_i(x) := \lambda x + (1-\lambda)p_i | i \in \{0,1,2,3\}\},$$

where $p_0 = (1,1,1), p_1 = (1,-1,-1), p_2 = (-1,1,-1), p_3 = (-1,-1,1)$ on $\mathbb{R}^3$. The p_i are the fixed points of the respective contractions, and Λ_λ lies in their convex hull Δ.

By definition the attractor Λ_λ satisfies

$$\Lambda_\lambda = \bigcup_{i=0}^{3} f_i(\Lambda_\lambda).$$

An approximation to the attractor Δ_n is formed inductively by iterating the IFS such that

$$\Delta_n = \bigcup_{\epsilon \in \sum^n} f_\epsilon(\Delta),$$

where $\epsilon = (\epsilon_0, \ldots, \epsilon_{n-1}) \in \sum^n$, $\sum = \{0,1,2,3\}$, and $f_\epsilon = f_{\epsilon_0} \ldots f_{\epsilon_{n-1}}$.

Since $f_i(\Delta) \subset \Delta$ it follows that $\Delta_{n+1} \subset \Delta_n$ and thus

$$\Lambda_\lambda = \lim_{n\to\infty} \Delta_n = \bigcap_{n=1}^{\infty} \Delta_n.$$

Each of the models in this paper was formed in either OpenSCAD in the case of 3D objects, or Maple 2015 for 2D objects by computing Δ_n for n between 5 and 7.

For $\lambda = 1/2$, Λ_λ is the standard Sierpiński tetrahedron. For values of $\lambda < 1/2$ the intersections $f_i(\Delta) \cap f_j(\Delta) = \emptyset$ for $i \neq j$ and thus Λ_λ is disconnected and therefore unprintable as a single object.

Golden Sierpiński Tetrahedra

Values of λ for which substantial overlaps occur, i.e., for which $f_i(\Delta) \cap f_j(\Delta)$ has interior points, were first considered by Broomhead *et al.* [3]. They show that for the Sierpiński triangle this property of overlap occurs for $\lambda > 1/2$ and that for certain values of λ the fractal is totally self-similar and has empty interior. They also generalize their results to 3 or more dimensions, although this paper is the first to present the attractors graphically and as physical models.

Definition. *We call any set S that satisfies $f_\epsilon(S) = f_\epsilon(\Delta) \cap S$ for any $\epsilon \in \sum^n$ and any n totally self-similar.*

Broomhead *et al.* were able to prove in [3] that for each of the unique positive solutions ω_m to the equation

$$\lambda^m + \lambda^{m-1} + \cdots + \lambda = 1 \tag{1}$$

the attractor Λ_{ω_m} is totally self-similar and has empty interior.

The totally self-similar property arises from the fact that for these values of λ the holes in the overlapping regions of the attractor perfectly align with each other. This alignment was found by solving for values of λ for which the overlap region is an image of Δ. Namely $f_i(\Delta) \cap f_j(\Delta) = f_i(f_j^m(\Delta))$ for each solution ω_m to equation (1). It is an interesting fact that for $m = 2$, $\omega_m = (\sqrt{5}-1)/2 \approx 0.618$, the reciprocal of the golden ratio. This gives rise to the use of the name Golden Sierpiński tetrahedra to describe the attractors Λ_{ω_m}. For $m = 2$ and $m = 3$ the 3D printed approximations to the attractors in $\mathbb{R}^3$ are shown at the top of the next page.

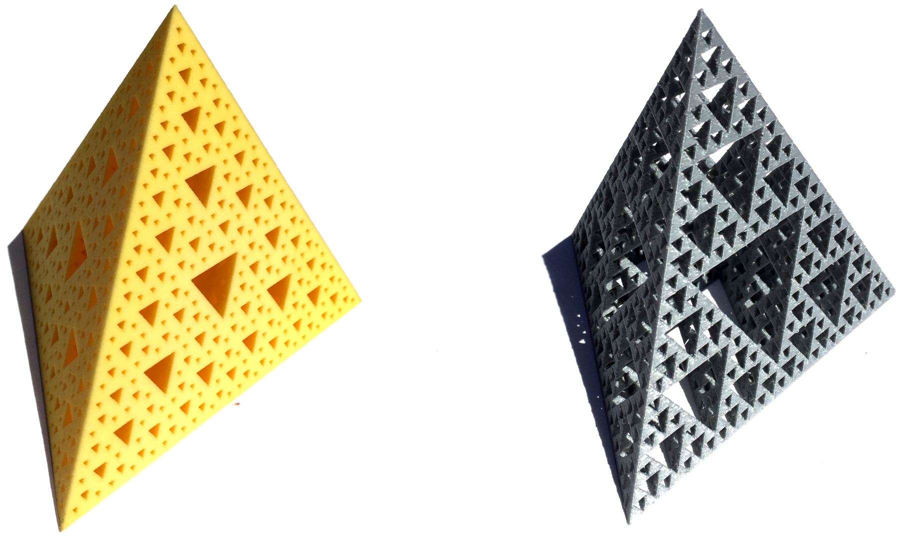

Figure 2: 3D printed approximations to Λ_{ω_m} for $\lambda = \omega_2 \approx 0.618$ and $\lambda = \omega_3 \approx 0.544$

Other Sierpiński N-gons with Overlap

The Sierpiński triangle can be naturally extended to form other fractals by starting with an IFS whose fixed points form higher order N-gons as Dennis and Schlicker have shown in [4]. In three dimensions this idea has been expanded to create fractals such as the Sierpiński Octahedron and the Menger Sponge. In the case of the Sierpiński Octahedron the fractal is structurally sound with a standard contraction rate given by $\lambda = 1/2$. Although 3D printable with this contraction rate we may extend the results of the Sierpiński Tetrahedron to this fractal to yield a model that is visually pleasing. Given a contraction rate of ω_2 we observe a similar pattern to the one formed in the tetrahedron. The attractor maintains the property of total self-similarity and its structural stability is further increased. This allows for the fractal to be formed without the risk of the individual components separating from each other during the process of 3D printing.

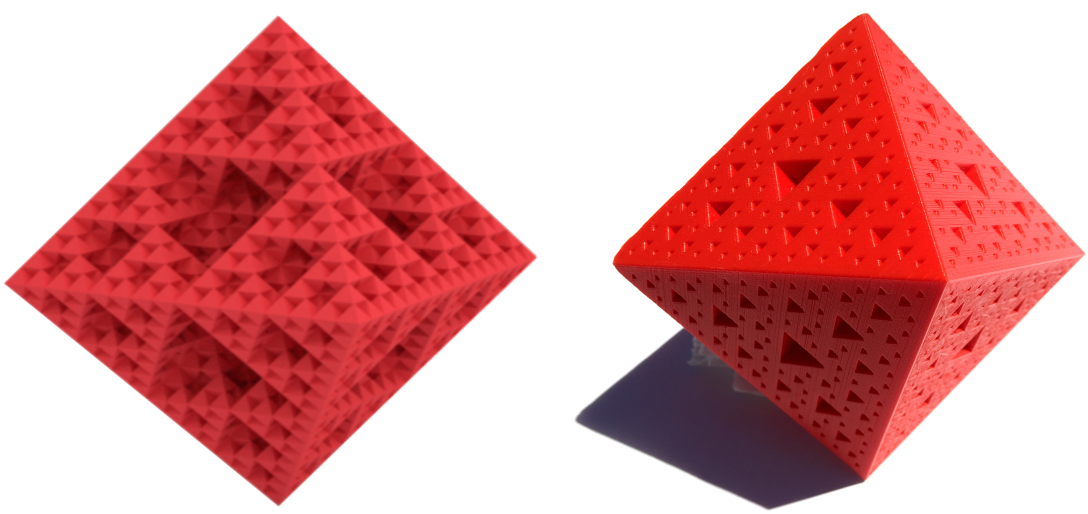

Figure 3: Approximation of the Sierpiński Octahedron with $\lambda = 1/2$ and a 3D printed version with $\lambda \approx \omega_2$

As was done with the Sierpiński Triangle to form the Sierpiński Tetrahedron, a similar generalization of the Sierpiński Carpet gives rise to the familiar Menger sponge. By increasing the contraction rate beyond $\lambda = 1/3$, so that there is significant overlap, the sizes of the holes in the fractal are reduced and therefore the integrity of the 3D printed model increases. It remains to be proven, but for each unique positive solution to

$$2\lambda^n + 2\lambda^{n-1} + \cdots + 2\lambda = 1 \tag{2}$$

the Menger Sponge appears to be totally self-similar. For $n = 2$ the solution to equation (2) is given by $\lambda = (\sqrt{3} - 1)/2 \approx 0.366$.

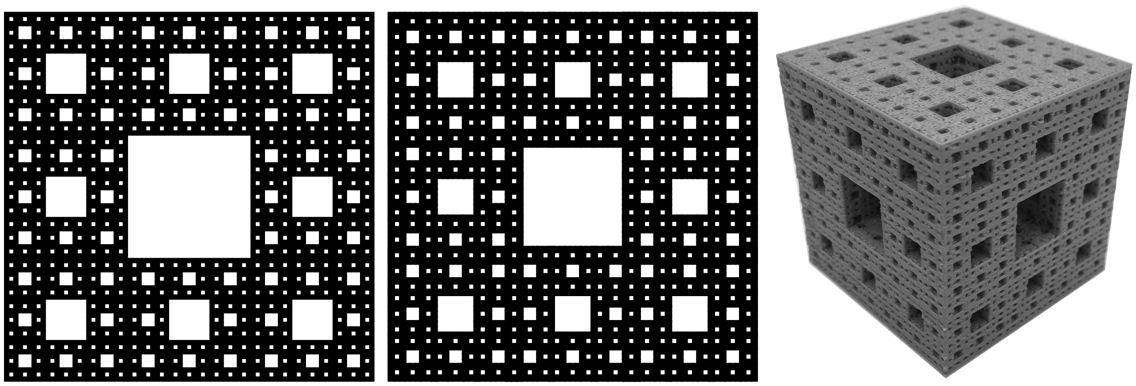

Figure 4: Sierpiński Carpet with $\lambda = 1/3$ and $\lambda = 0.366$, and a 3D printed Menger Sponge with $\lambda = 0.366$

In a similar way to that which Broomhead *et al.* used to find equation (1) in [3], I solved for values of λ for which the holes in the overlapping regions of the Menger Sponge perfectly align with each other. This process gave rise to equation (2). By looking closely at Figure 4 you can observe that the number of holes in the Sierpiński Carpet when $\lambda = 0.366$ is reduced compared to when $\lambda = 1/3$. This is precisely due to the alignment of the smallest holes shown in the overlapping regions of the attractor.

Conclusion

The methods shown here can be further applied to other fractals in three dimensions to yield new models that may be manufactured by 3D printing. For example, by starting with an IFS that has as a convex hull, a dodecahedron, the dodecahedron flake is formed. By increasing the contraction rate of such attractors beyond the point where the functions in the IFS are "just touching" to where they have significant overlap we can create fractals that are conducive to 3D printing. Further work remains to find specific contraction rates for which the resulting fractals have significant overlap and are also totally self-similar.

References

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[5] B. Hasselblatt and D. Plante, "On the Interior of "Fat" Sierpiński Triangle", *Experimental Mathematics*, 23:3 (2014), pp. 285–309.