Quilting the Klein Quartic

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Abstract

The Klein Quartic curve contains the maximal number of symmetries a genus 3 surface can have: $84(g-1) = 168$. We create a fabric model of 24 regular heptagons that not only captures the platonic nature of the Klein Quartic, but it is flexible enough to be everted through any of its holes, thus illustrating 24 of the 168 symmetries, whilst rigid models can only display 12.

Figure 1: Four views of the Klein Quartic Quilt.

The Klein quartic has long captured the imaginations of mathematicians and artists, most notably with the Helaman Ferguson sculpture *The Eightfold Way* at MSRI.

Its natural structure, given by $x^3y + y^3z + z^3x = 0$, lives in $\mathbb{C}P^2$ [5]. It is difficult to imagine surfaces in this space, but it is covered by a tiling in $\mathbb{H}^2$ that shows the symmetries of this object. In order to go from its universal cover in $\mathbb{H}^2$ to a compact surface in $\mathbb{R}^3$, sides of the fundamental 14-gon need to be glued up, as shown in Figure 2. After these identifications have been made, one finds a remarkable symmetry in traversing the surface around any of its handles. One can trace out an *eightfold way* – a closed path through eight heptagonal edges made by turning alternately left and right. Every gluing connects two ends of a path. (See Figure 2.)

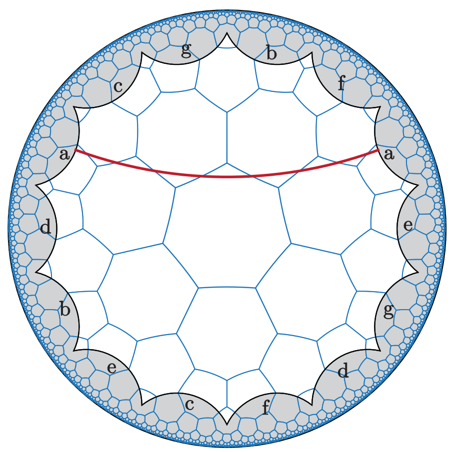

Figure 2: Klein's fundamental domain in the Poincaré disk model, and an eightfold way winding through the surface.

The Klein quartic is *platonic* surface – its symmetry group acts transitively on flags of faces, edges and vertices [4]. The

(a) Each pair of pants is made from six regular heptagons.

(b) Three pairs of pants are joined around the central one (orange, navy on black and tan with leaves).

(c) One possible pair of pants decomposition.

Figure 3: The Klein Quartic contains four pairs of pants, each joined to the other via an eightfold way.

Klein quartic is a Riemann surface of genus three, and its symmetry group can be captured in the $\{7, 3\}$ tiling of the hyperbolic plane. In this setting, one can see all 168 automorphisms [6, 8]. The symmetry group is transitive on the heptagons – each of the 24 heptagons can be interchanged with any other heptagon and then it can be rotated by any one of seven $2\pi/7$ rotations, yielding a symmetry group of order $24 \times 7 = 168$ [1] – the maximal number of symmetries possible in a genus three surface [3].

With Klein's Quartic Quilt, we wanted to create a sculpture that captured the subgroup with as many symmetries as possible in the mapping of the Klein quartic into $\mathbb{R}^3$. The inspiration for our construction came from a post by Mike Stay [10] and the quilt by Eveline Séquin [9]. The Klein quartic admits a *pair of pants* decomposition – a decomposition into pieces of surface each homeomorphic to a three-holed sphere. Here, four groups of six regular heptagons can be joined along eightfold ways to form the Klein quartic [4] (see Figure 3).

Fabric is a wonderful medium because it is flexible enough that we can evert the *tetrus* – an embedding of the three-handled torus $\mathbb{R}^3$ with tetrahedral symmetry. This eversion, as in the animation by Greg Egan [2], shows a symmetry subgroup of order 24 [1]. The additional six rotations by $2\pi/7$ of each of the heptagons could theoretically be realized, but for the physical limitations of the eversion process. The seams cannot pass through one another, which leaves the orientation of each of the heptagons fixed with respect to their position on the tetrus. In order to highlight the fact that this is a platonic surface that can be tiled either by 24 heptagons, three meeting at each vertex, or by 56 equilateral triangles, seven meeting at each vertex, we have constructed the quilt blocks out of heptagons and used gold thread to quilt the dual tiling.

The list of symmetries of the Klein Quartic Quilt:

- The identity element.
- Order two rotations by π about the midpoint of any edge joining two pairs of pants. There are three of these symmetries. (See Figure 4)

Figure 4: Ignoring the colouring, these figures illustrate two-fold symmetry about the midpoint of edges joining two pairs of pants. The axis is out of the page.

[1] Including the colourings of the fabrics in the elements of the symmetry group of the Klein Quartic Quilt yields only the identity.

• Order three rotations by $2\pi/3$ and $4\pi/3$ about each of the vertices of the tetrahedron. There are two rotations about each of the four axes for a total of eight of these symmetries. (See Figure 5)

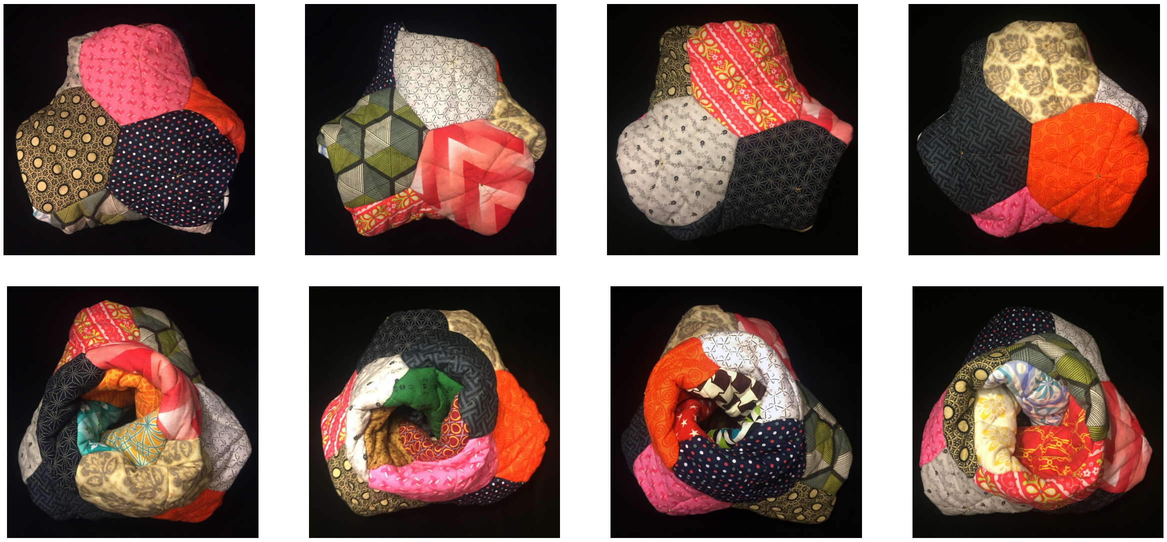

Figure 5: Images from the two ends of the three-fold symmetry axes. There are eight $2\pi/3$ rotations in the Klein Quartic Quilt.

• Eversion of the tetrahedral model of the Klein Quartic is an order two operation - turning it inside out through any of its holes. This is a rotation of the surface along an eightfold way. There is one of these symmetries.

Figure 6: Eversion of the surface by pulling it inside out through the central hole.

• Composition of eversion and rotation by π about the midpoint of any edge joining two pairs of pants is an order two operation. There are three of these symmetries. (See Figure 7)

Figure 7: Eversion followed by two fold rotations about the midpoint of edges joining two pairs of pants.

• Composition of eversion and rotation by $2\pi/3$ or $4\pi/3$ about each of the faces of the tetrahedron is an order six operation. There are eight of these symmetries. (See Figure 8)

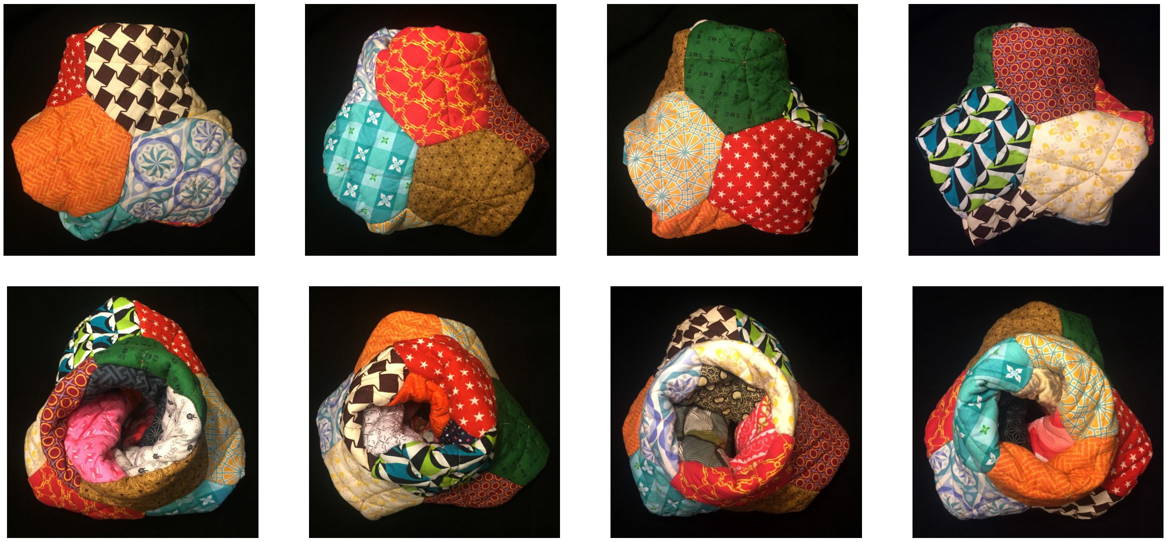

Figure 8: Images of the two ends of the three-fold rotation axes after eversion.

A tiling of tilings in Klein's Quartic Quilt

The $\{7,3\}$ heptagonal tiling of the Klein quartic shows its hyperbolic nature. Yet the Klein Quartic Quilt is made from fabric, which is euclidean. All fabric patterns, therefore, must be elements of a wallpaper group. All 17 of the planar symmetry groups are represented in the fabric prints on chosen for the Klein Quartic Quilt. Yet there are 24 heptagons in the Klein quartic. Can you spot the duplicates?

References

[1] John Baez. Klein's Quartic Curve. Available at http://math.ucr.edu/home/baez/klein.html., 2013.

[2] Greg Egan. Klein's Quartic Curve. Available at http://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuartic.html., 2006.

[3] A. Hurwitz. *Über algebraische Gebilde mit eindeutigen Transformationen in sich*.

[4] Hermann Karcher and Mathias Weber. The Geometry of Klein's Riemannian Manifold. In Silvio Levy, editor, *The Eightfold Way: The Beauty of Kleins Quartic Curve*, pages 9–49. Cambridge Univ. Press, 1999.

[5] Felix Klein. *Über die Transformationen siebenter Ordnung der elliptischen Funktionen*. *Math. Annalen*, 14:428–471, 1879. (Translation into English by S. Levy [7].).

[6] J. Lehner and M. Newman. On Riemann surfaces with maximal automorphism groups. *Glasgow Math. J.*, 8:102–112, 1967.

[7] Silvio Levy. *The Eightfold Way: The Beauty of Kleins Quartic Curve*. Cambridge Univ. Press, 1999.

[8] H. E. Rauch and J. Lewittes. The Riemann surface of Klein with 168 automorphisms. In R. C. Gunning, editor, *Problems in Analysis*, pages 297–308. Princeton Univ. Press, 1970.

[9] Carlo Séquin. Patterns on the Genus-3 Klein Quartic. In *Proceedings of Bridges 2006: Mathematics, Music, Art, Architecture, Culture*. Tessellations Publishing, 2006.

[10] Mike Stay. Klein's Quartic Curve. Available at http://math.ucr.edu/~mike/klein/. Accessed on 26 Feb 2017 at 21:19:29.