Combinatorics in the Art of the Twentieth Century

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Abstract

This paper is motivated by a question I asked myself: How can combinatorial structures be used in a work of art? Immediately, other questions arose: Whether there are artists that work or think combinatorially? If so, what works have they produced in this way? What are the similarities and differences between art works produced using combinatorics? This paper presents the first results of the attempt to answer these questions, being a survey of a selection of works that use or contain combinatorics in some way, including music, literature and visual arts, focusing on the twentieth century.

Introduction

Combinatorics is the study of sets, subsets and other discrete, finite structures, as defined by a number of accepted combinatorial rules. Typical combinatorial problems are existence, counting, and optimisation of a particular configuration. Combinatorics is related to many other branches of mathematics, for instance, enumerative combinatorics, or the art of counting, is the basis of (discrete) probability. That is to say, it is related to randomness in the sense that whenever you have a choice to make, the study of the range of that choice is a combinatorial problem. The basic configurations are the permutations and combinations of the elements of a set. Other combinatorial structures include graphs, hypergraphs, partitions of sets, and combinatorial designs which are much more complex, including Latin squares and finite geometries, and have fundamental connections with finite fields, group theory and number theory. (See Cameron [6] and Stinson [25].)

Motivated by the question: "How can combinatorial structures be used in a work of art?", this paper presents a survey of the relationship between combinatorics and the arts, focusing on the twentieth century. The emphasis is on artists that take a combinatorial approach to their work, and on works in which combinatorial structures are part of the creative process itself rather than simply being a tool to achieve some kind of 'artistic' result. In such works, whether the artist intends to make the combinatoric structure visible or not, it is consciously and purposely made part of the work rather than being only a means to an end.

The next three sections present the research into combinatorics in music, in literature, and in visual art, respectively. Each of these three sections begin by establishing the background, then go on to offer a description of the combinatorial work of a particular artist or group of artists. The selection of the artists is a personal choice by the author in an attempt to demonstrate the different ways in which combinatorics manifest across disciplines.

Combinatorics in Music

Music has been associated with mathematics from its beginnings. Among the mathematical models that have inspired musical theories, combinatorics has had a continuous role because almost all musical systems are discrete. From the musical systems of the Middle Ages to the present day, combinatorics has been present, underlying scales and chords, and compositional techniques, as explained by Knoblock [12] and Nolan [18]. Benson [2] dedicates a chapter to the study of symmetry in music, which includes a section on change ringing and permutation groups, and another on the use of Pólya's enumeration theorem in solving music-related counting problems. Read [21] presents a number of musical combinatorial problems, with a focus on Messiaen and Stockhausen. Sethares [24] carried out an original study of rhythm, with a chapter dedicated to its combinatorial aspects. In fact, when composition is viewed as a methodology of choice, the range of options that a musical system provides is always a combinatorial problem (see chapter 9 of Loy [15]).

In the twentieth century, just as the conventions of figurative painting were radically transformed by abstraction, so too the fundamental forms of music were changed by the shift from tonal music to atonality. Musicians who wished to break with tradition began to explore new sonic territories, for which they needed new methodologies. This new way of thinking resulted in the stronger presence of combinatorics in modern musical works.

Dodecaphonic music. In addition to these new developments, there emerged dodecaphonic music, also known as serialism, developed by Arnold Schoenberg [23]. This methodology of composition, based on the permutations of the 12 notes of the chromatic scale, was later transformed into a more organised system, so-called "integral serialism".

The first wave of serialist composers, the Second Viennese School, was a group of composers from the early twentieth century formed by Arnold Schoenberg and his pupils, including Alban Berg and Anton Webern among others. With the objective of deconstructing tonal expectation, serialism works with 12-tone rows, consisting of ordered sequences of the 12 notes in which each of the notes appears just once. Schoenberg established the following rules, which he called *Method of Composing with Twelve Tones Which are Related Only With One Another* [23]:

- A composition uses a particular ordering of the 12 notes. The chosen tone row is called the *basic set*. It can be viewed both as a sequence of notes or a sequence of intervals, and it can be denoted by a permutation of the set $\{0, 1, \ldots, 11\}$. The notes can be freely applied to any octave.
- Each time a new tone is needed in the composition, the composer uses the next tone in the row, circling back to the first tone when the row is exhausted.
- The composer can also use the tone rows obtained from the basic set via the application of the following operations:
  - **Inversion.** Replace each tone by its negative (modulo 12).
  - **Retrograde.** Read the sequence in reverse order.
  - **Retrograde inversion.** The composition of the inversion and the retrograde operations.
  - **Transposition** by n half-tones. Add n to each tone (modulo 12).

After World War II, a second group of composers, including Pierre Boulez and Milton Babbitt, inspired by the systematic treatment of pitch, rhythm, dynamics and articulation of the compositions of Anton Webern, created a new form of music. Integral serialism is a composition methodology that represents a stage further in dodecaphonic techniques. Essentially, it consists of extending the 12-tone ordering technique to all other parameters of music. For instance, Olivier Messiaen established series or modes of 36 pitches, 24 durations, 12 articulations or attacks, and seven degrees of loudness for his work *Mode de valeurs et d'intensités*, a work that was not strictly dodecaphonic but inspired, as Anton Webern did, this second group of composers.

Tom Johnson. Another composer that use combinatorics in his work is Tom Johnson [7], who bases his work on mathematical ideas, with especial use of combinatorics, taking an approach quite different from that of the serialist composers. Johnson uses combinatorics to define the structure of his compositions, sometimes using complex combinatorial design configurations. However, as he himself admits, one of his goals is to make his music accessible, understandable, and its mathematical (combinatorial) structure and process easy to perceive. He has sometimes been called "The man who counts".

Johnson devises graphs and diagrams to represent and understand the structures used in his compositions. He collaborated with the mathematician Franck Jedrzejewski to explain the combinatorial content of his drawings and music in *Looking at Numbers* [11].

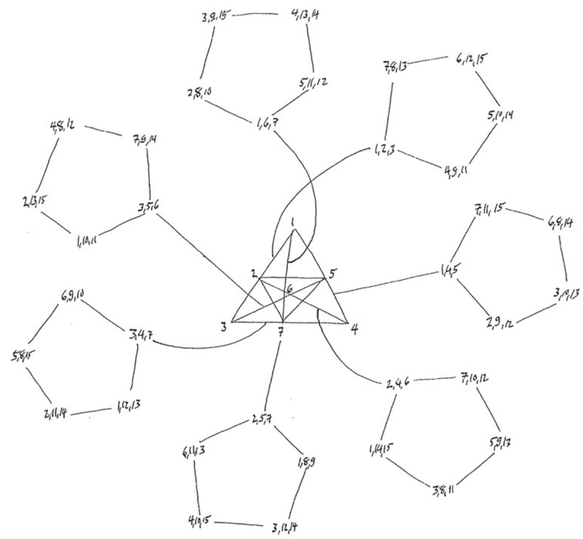


Figure 1: *A representation of the* $(15, 3, 1)$ *design used by Tom Johnson in* Kirkman's Ladies. *(With permission of the author.)*

A selection of techniques used in Johnson's combinatorial works from the early 2000s include the following:

- **Integer partitions** are used to construct chords with the same average height. In *Trio*, the three instruments play chords that add up to 72 while in *Hexagons*, two sums, 30 and 31, are combined.
- **Subsets.** In *Mocking*, three percussionists play rhythms constructed by choosing 4 different numbers from 1 to 8. This is represented by Johnson as sums from $10 = 1 + 2 + 3 + 4$ to $26 = 5 + 6 + 7 + 8$, in a graph connecting intersecting subsets.
- **Combinatorial designs.** Working with block designs, Johnson identifies the elements of the blocks with musical parameters, such as pitches and rhythms, and uses the combinatorial structure of these blocks to create a path through the musical material, linking blocks by their common objects.

  In *Vermont Rhythms*, he uses 42×11 rhythms based on the $(11, 6, 3)$ design, while *Block Design for Piano* is built on the 4-$(12, 6, 10)$ design defined by 30 base blocks and one automorphism of the permutation group over 12 elements. The score *Kirkman's Ladies* was inspired by the well known *school girls problem*, posed by Reverend Kirkman in 1847 in *The Lady's and Gentleman's Diary*:

  > "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."

  The solution to this problem involves Kirkman triple systems, a special case of combinatorial design. In the score, Johnson uses a $(15, 3, 1)$ design with 13×35 blocks (see Figure 1). As he explains:

  > "In my score, entitled Kirkman's Ladies, the 15 ladies become a scale of 15 notes, and the daily walks of five rows, three ladies in each row, become phrases of five chords with three notes in each chord. Each lady/note occurs once in each sequence of five chords, each pair of ladies walks together once a week, and by the end of the 13 weeks/sections, all 455 possible trios of women, all 455 possible combinations of three notes, have passed by." (See [11], page 39.)

Combinatorics in Literature

In literature, we have to distinguish poetry from prose since metrics and rhyme patterns can be considered combinatorial structures. Moreover, mathematical permutations have been used in poetry since the twelfth century. The sextine, invented by the troubadour Arnaut Daniel, is a famous structure that involves a permutation of 6 elements. Another known poetic form that uses permutations is the proteus poem, in which each line employs the same words but in a different order. The earliest surviving proteus poem dates back to 1561 (see Higgins [10]).

In the twentieth century, we find permutations in the cut-up technique of the Dadaists, and the works of digital poetry pioneer Brion Gysin, among others (see Funkhouser [8]). In prose, the innovative narrative techniques of the Argentinian writer Julio Cortázar eschew temporal linearity. His most famous work is the novel *Rayuela*, in which the reader decides the chapters' reading order. Another work is the novel *Composition no. 1*, by the French experimental writer Marc Saporta, where pages are read in arbitrary order.

The "Ouvrier de Littérature Potentielle", Oulipo, was a group of writers and mathematicians founded in 1960 by François Le Lionnais and Raymond Queneau (both mathematicians) as a reaction against both traditional, in particular romantic, literature and surrealism. The group worked basically with two objectives: first, to generate what they called "potentiality", to provide methodologies to create potential works, even if they were not realised, to create what they called "the book of all stories"; and second, to write with "constraints" or rules, in order to elaborate new forms and structures to serve as a support for literary works. This technicist vision of literature, aiming to use language in a more abstract manner and understand it as a sort of Meccano of signs which can be assembled by following rules, combinations, or even algorithms, is what gives to the works of the Oulipo their combinatorial character. Claude Berge, a member of the Oulipo and, as a mathematician, a renowned expert in combinatorics and graph theory, explains in [3] how the literature of the Oulipo is related to combinatorics and proposes many ways to represent the combinatorial structure of some Oulipian works using graphs. The following authors produced works as part of this group:

Raymond Queneau. His work *Cent mille milliards de poèmes* (1961) is, in fact, a set of 10 sonnets. Since each sonnet follows the same rhyme scheme and bears the same rhyming sounds in the same places, each line of one poem can be replaced by the equivalent line of another without breaking any formal rules. The work is made by cutting slits on either side of each line of each poem, which yields a set of 140 recombinable lines, easy to rearrange into one hundred trillion possible sonnets. In this way, the author goes beyond the use of a permutation, with the purpose of having the whole set of poems together, showing the potentiality of the work.

George Perec. *La vie, mode d'emploi* (1978) is an extremely detailed description of the rooms of a house and of everything found there, including occupants, at a particular moment of time. Throughout the novel, many connections are established between the rooms, objects and characters, which create a network, a novel of novels. A number of combinatorial schemes are involved in the writing of *La vie, mode d'emploi*. For example, the rooms are mapped onto an orthogonal Latin square of order 10, which consists of two Latin squares over two n-sets, S and T, defined over the same $n \times n$ square, so that each pair (s, t) of the Cartesian product $S \times T$ appears exactly once. Furthermore, during the first part of the novel, the movements around the house, that is, the 10×10 square, follow a hamiltonian path, according to the movements of a knight on a 10×10 chessboard [19].

Italo Calvino. Calvino, an Italian writer who joined the Oulipo in the last stage of his career, wrote novels that potentially contain many different stories which must be constructed by the reader, and that have been called "combinatorial literature" [5]. As a precursor of hypertext, he called his novels "hypernovels". In this period, the most famous books he wrote are *Le città invisibili* (1972), a series of short stories that describe a number of fantastic cities; *Il castello dei destini incrociati* (1973), for which he used combinations of Tarot cards; and *Se una notte d'inverno un viaggiatore* (1979), a collection of interwoven stories, some of them interrupted by accidents such as an error of typing (purposely made by the author).

Combinatorics in Visual Art

The interaction between art and geometry has existed since antiquity. In the early decades of the twentieth century, starting with the experiments in Cubism and Expressionism, visual art turned from figurative to abstract. A large number of artistic movements arose, some of them inspired by mathematics and, in the case of Constructivism, Geometric Abstraction and Minimalism, by the most simple forms of geometry. Non-representational compositions were concerned with the production of various geometric shapes where the size and character of these shapes, their relationship to each other, as well as the colours used throughout the work, become the defining motifs of abstraction. Such art works were conceived as combinations of forms and colours. Lorenzi and Francaviglia's survey [14] provides detailed descriptions of the relations between geometry and art in the twentieth century.

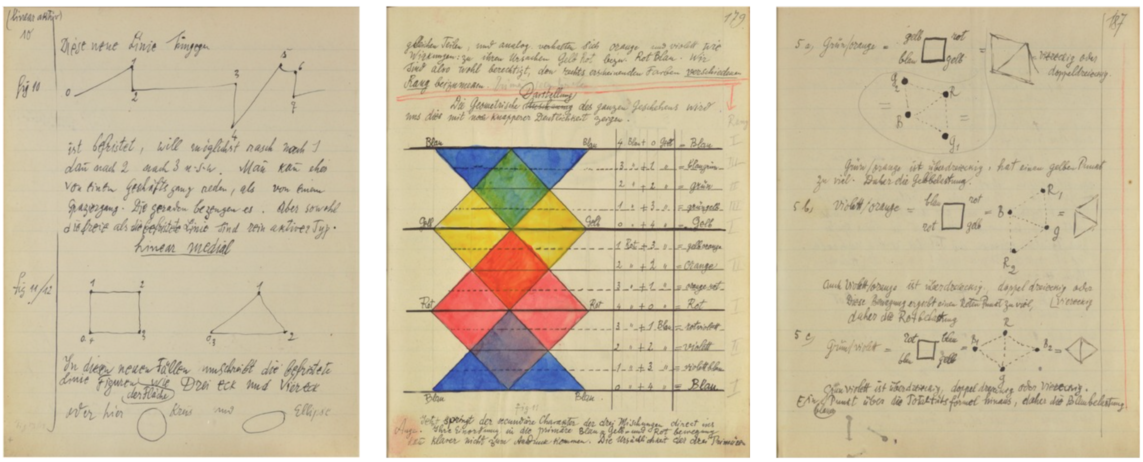


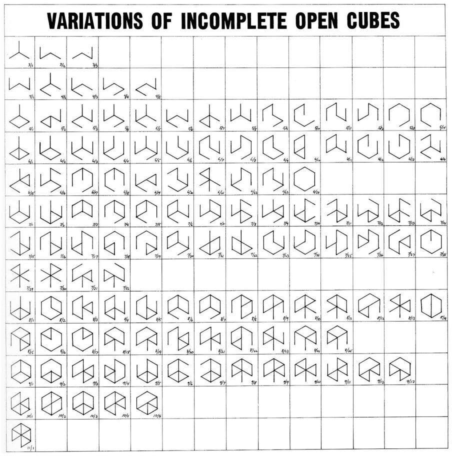
Figure 2: *Three pages of Paul Klee's* Teaching Notes on Pictorial Creation *he taught at the Bauhaus from 1921 to 1931. (From the Online-Data Base at Zentrum Paul Klee, public domain.)*

In some cases, geometry is the visual part of a work that makes use of, or is based on, combinatorial structures and methodologies. In this sense, and despite the strong geometric character of the works of avant-garde artists, Paul Klee, Vassily Kandinsky, and Josef Albers, who were teachers at the Bauhaus in the interwar period, were influenced by the work of the chemist and Nobel laureate Wilhelm Ostwald, who used combinatorics as a creative and interdisciplinary way of thinking in areas such as knowledge organisation and in his theory of colours and forms, as studied by Hapke [9]. In Figure 2, three pages of the teaching notes of Paul Klee at those years show how he used graphs to represent the relations between colours.

The following three artists – Sol LeWitt, Vera Molnar, and Manfred Mohr – show three different approaches to combinatorics in their art.

Sol LeWitt. A conceptual artist, LeWitt is known for his series of cube works and for his wall drawings. His first cube work, *Serial Project #1* (1966), shows the 36 possible configurations that result from two cubes nested into each other, each with two parameters, "surface" (open/closed) and height (low/middle/high). In his accompanying text [13], LeWitt describes the combinatorial rules that define this work. *Variations of incomplete cubes* (1974), shown in Figure 3, is another combinatorial work: the construction of all the possible figures that can be obtained by removing the edges of a cube.

Other works that follow combinatorial rules are *Wall Drawing #450, A wall is divided vertically into four equal parts. All one-, two-, three- and four-part combinations of four colors* (1985) and *Wall Drawing #493, The wall is divided vertically into three equal parts. All one-, two-, and three-part combinations of three colors* (1986).



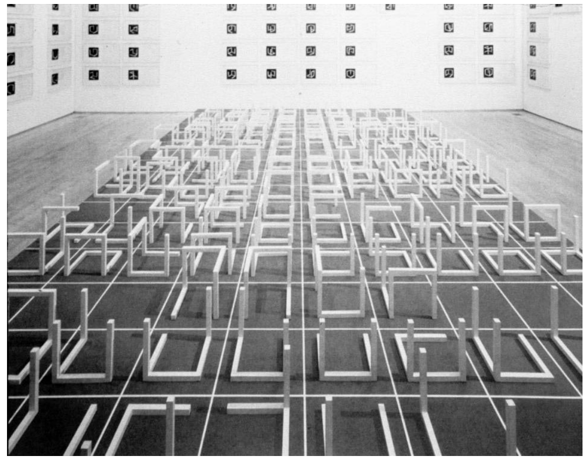


Figure 3: *Sol LeWitt's* Variations of incomplete open cubes *(1974). (From* `http://socks-studio.com`*, under Creative Commons Attribution-ShareAlike 3.0 license.)*

LeWitt's interest in making the combinatorial structure of the work visible to the observer contrasts with the conception of Donald Judd, an artist that worked in numerical sequences and whose early work influenced LeWitt. Judd thought that the mathematics in his work had to be somehow hidden or, at least, made less clear to the observer (Rottman [22].)

Vera Molnar. Vera Molnar [26], who was educated in classical painting, worked with a variety of techniques such as collage, gouache, pencil, and yet her fame comes from being a pioneer of digital art. She used the computer as a tool to develop a systematic language. "Its immense combinatorial capacity facilitates the systematic investigation of the infinite field of possibilities", she said [20].

In her work, influenced by the ideas of Max Bill [4] concerning mathematics in art, she used geometric figures such as squares, circles and lines, with an emphasis on repetition. Besides the use of simple geometric figures (as she says, "I love squares"), her art works are based on an investigation of the expressive possibilities of the contrast between order and disorder, which she accomplishes by combining irregularities or randomness with the structure given by combinatorics.

The representation and decomposition of the 3×3 grid, sometimes using polyominoes, is a recurrent theme in Molnar's work, and more generally, the square grid, a combinatorial structure that is decomposed and deconstructed in different ways in a large number of her works. One of these works is *Hommage à Dürer*, a series that she started in 1948, and on which she worked for more than 50 years. It represents a magic square by a line following a permutation of the squares of a 4×4 grid, referring to the magic square in the painting *Melencolia I*, by Albrecht Drer. A number of different permutations are generated and placed to occupy the squares of a 10×10, 15×15, or 20×20 grid.

Manfred Mohr. Manfred Mohr [16] was attracted to computer-generated algorithmic geometry after discovering Max Bense's information aesthetics [1] in the early 1960s. In 1969 he programmed his first computer drawings. Since then he has continued to develop and write algorithms for his visual ideas and his art is produced exclusively with computers. As he explains:

> "The first step in that direction was an extended analysis of my own paintings and drawings from the last ten years. It resulted in a surprisingly large amount of regularities, determined of course by my particular aesthetical sense, through which I was able to establish a number of basic elements that amounted to a rudimentary syntax. After representing these basic constructions through a mathematical formalism, and setting them up in an abstract combinatorial framework, I was in a position to realize all possible representations of my algorithms." (See [17], page 36.)

In 1973, Mohr started his work on the cube, which he has continued to the present day with multidimensional hypercubes and their combinatorial properties. Figure 4 shows three images of *Subsets*, one of Mohr's cube works. The algorithm selects a subset of cubes from a repertoire of 42,240 cubes inherent in the 11-dimensional hypercube and then decides which sides shall be black or white.

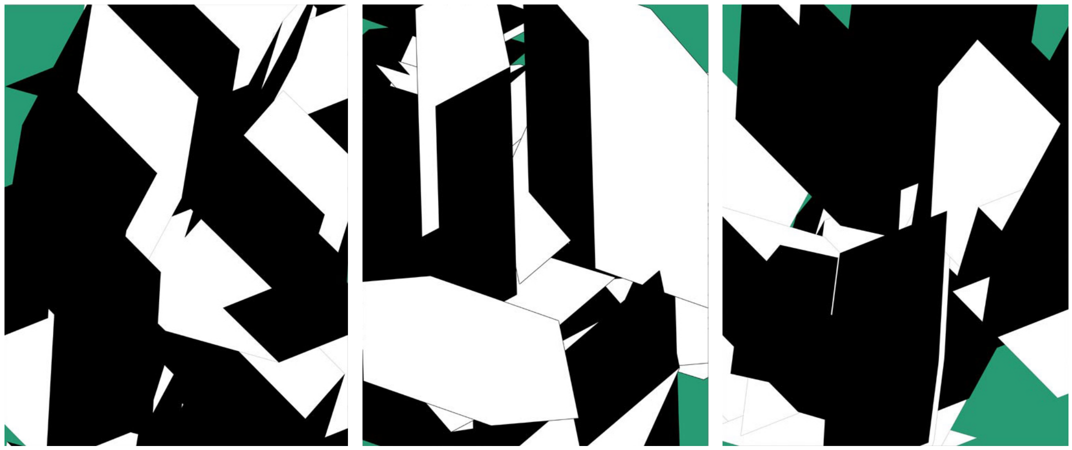


Figure 4 : Subsets *by Manfred Mohr (2003-2005), based on the* 11 *dimensional hypercube . (With permission of the author.)*

Conclusions and Further Work

This presentation of the combinatorial work of a selection of artists from the twentieth century forms a necessarily brief survey. A more extensive paper would be needed to present a deeper analysis. In more detail:

- The works of Sol LeWitt, Vera Molnar and Manfred Mohr deserve more attention, and elements of the research conducted could not be presented here due to lack of space.
- Other musicians that make use of combinatorics include Elliot Carter, Iannis Xenakis, and more recently, Bryn Harrison and Samuel Vriezen, to cite just a few. The study of how these and other musicians work with combinatorics is part of a planned future research project, including musicians of other genres, such as jazz, computer music and math rock.
- A more extensive investigation of the use of combinatorics in literature, including digital poetry, is also planned.
- In the visual arts, studies have already commenced on the combinatorial nature of the works of artists other than the ones presented here, including Victor Vasarely, Bridget Riley, François Morellet, Luis Tomasello, Mary Martin, Gego, and the new media artists Vladimir Bonačić, Hiroshi Kawano, Rockne Krebs, and Edward Zajec.
- Future plans also include an exploration of how combinatorics is used in films such as John Whitney's *Permutations*, or the works by the experimental filmmaker Raul Ruiz.

This project is a work in process. Through the exploration and analysis of artists working with combinatorics in several disciplines, the aim is to answer, if possible, some of the questions that arose in the first stage of research: Can an artist be a combinatorial artist? Can we say that an artist thinks combinatorially? Also, combinatorics seems to be related to works based on rules or structure, such as conceptual art and generative art which begs the further question: Is that the only way combinatorics can be related to art?

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