The Pentagonal Numbers Meet the Choose-4 Numbers

James Morrow
Mathematics Department
Mount Holyoke College
South Hadley, MA, 01075, USA
E-mail: jmorrow@mtholyoke.edu

Abstract

The artistic endeavor of visually representing 70 as both a pentagonal number and a combinatorial number raised some questions, conjectures, and then proofs, concerning the relationship between pentagonal numbers and Choose-4 numbers in Pascal's triangle. The proofs, initially failing to give *understanding*, inspired, through visualization, a deeper mathematical understanding and an interesting and satisfying solution to the initial artistic endeavor.

Introduction. This paper describes how an artistic problem inspired a mathematical problem, which, in turn, inspired a new artistic problem, a problem of finding an illuminating proof, and a solution to the original artistic problem. The artistic problem, motivated by a friend's 70^{th} birthday, was to visually represent 70 in two of its guises: a *pentagonal* number and a *combinatorial* number. This problem led to a mathematical quest to understand the relationship between the entire sequence of pentagonal numbers and the sequence of *Choose-4* combinatorial numbers. These two sequences are described below.

Pentagonal numbers. Pentagonal numbers, defined by: $P(1)=1$ and $P(n)=P(n-1)+3n-2$ for $n>1$, form a sequence, 1, 5, 12, 22, 35, 51, 70, … . The pentagonal sequence is a *figurate* sequence, meaning that you can visualize it by using polygons. In the pentagon to the right, the single dot at the top represents $P(1)=1$, the pentagonal number $P(2)=5$ is represented by the necklace of 1+4 dots, and each successive pentagon is constructed from the previous pentagon and an additional necklace of dots. *Triangular* numbers, a figurate sequence based on a triangle of dots and defined by $T(1)=1$ and $T(n)=T(n-1)+n$, are also used in this paper.

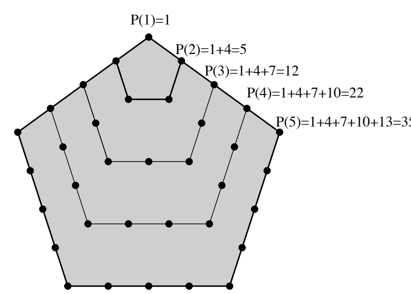


***Figure 1**: The definition and first five pentagonal numbers, pictured as a sum of dots*

Combinatorial numbers. Combinatorial numbers tell how many combinations there are of n objects taken m at a time; the notation C^m represents this number of combinations taken m at a time. Importantly for this paper, the first Choose-4 number, $C^4(1)=1$, is the number of combinations of 4 objects taken 4 at a time, the second Choose-4 number, $C^4(2)=5$, is the number of combinations of 5 objects taken 4 at a time, and so on. All of the combinatorial numbers appear in Pascal's Triangle.

A diagram to compare pentagonals and Choose-4's. To see how the pentagonals and Choose-4's are related, consider Figure 2, which is Pascal's Triangle, augmented by a diagonal of pentagonal numbers on the right and an equation for $C^4(n)$. The Choose-4 numbers appear in the diagonal headed by $\boldsymbol{C^4(n)}$: 1, 5, 15, 35, 70, … Note that, in Figure 2, there is a diagonal of:

a. Choose-3's (starting to the right of the Choose-4's): 1, 4, 10, 20, 35, …
b. Choose-2's, also called triangulars, (starting to the right of the Choose-3's): 1, 3, 6, 10, 15, …

1

1 1

1 2 1

$C^3(1)+C^3(2)+C^3(3)+\ldots+C^3(n)=\boldsymbol{C^4(n)}$ 1 3 3 1 $\boldsymbol{P(n)}=1+4+7+\ldots+3n-2$

1 4 6 4 1 1 = $\boldsymbol{P(1)=P(P(1))}$

1 5 10 10 5 1 5 = $\boldsymbol{P(2)=P(P(1)+1)}$

1 6 15 20 15 6 1 12 = $P(3)$

1 7 21 35 35 21 7 1 22 = $P(4)$

1 8 28 56 70 56 28 8 1 35 = $\boldsymbol{P(5)=P(P(2))}$

1 9 36 84 126 126 84 36 9 1 51=$P(6)$

1 10 45 120 210 252 210 120 45 10 1 70 = $\boldsymbol{P(7)=P(P(2)+2)}$

1 11 55 165 330 462 462 330 165 55 11 1 92=$P(8)$

1 12 66 220 495 792 924 792 495 220 66 12 1 117 = $P(9)$

Figure 2*: Pascal's Triangle, with equations for $C^4(n)$ and $P(n)$ and pentagonals appended on the right*

Patterns in Figure 2. Looking at Figure 2 and calculating $P(n)$ and $C^4(n)$ for many n, strong patterns emerge: There are lots of pentagonal numbers that are Choose-4 numbers, and, even more striking, exactly two of every three consecutive Choose-4 numbers are pentagonal. The forms for the pentagonals that are also Choose-4's are somewhat strange looking: They are either of the form $P(P(n)$ (the $(P(n))^{th}$ pentagonal number), or $P(P(n)+n)$ (the $(P(n)+n)^{th}$ pentagonal number). The forms for those Choose-4 and pentagonal numbers appear in bold in Figure 2. With much calculation, organization, and observation, I conjectured to be true and then proved to be true, that for all positive integers n:

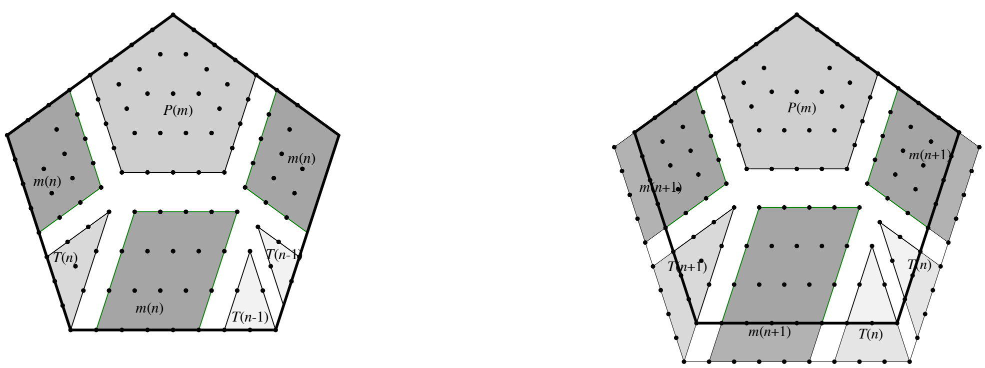
Theorem 1: $C^4(3n-2) = P(P(n))$
Theorem 2: $C^4(3n-1) = P(P(n)+n)$
Theorem 3: $P(P(n)+2n) < C^4(3n) < P(P(n)+2n+1)$

Interpretation of the theorems. Note that the theorems include all possibilities for $C^4(m)$, because each integer m is exactly one of the expressions $3n$-2, $3n$-1, or $3n$ for some integer n. Thus you know where in the pentagonal sequence every Choose-4, $C^4(m)$, appears: Where the pentagonal is of the form $P(P(n))$, $P(P(n)+n)$ or between the two consecutive pentagonals $P(P(n)+2n)$ and $P(P(n)+2n+1)$, depending on whether m is equivalent, modulo 3, to 1, 2, or 3. This fact is to me both surprising and beautiful.

Even before I proved that the three statements are true, I was virtually *certain of their truth*, because the pattern of occurrences is so strange, yet easily verifiable for specific integers. (I went as far as $C^4(16)=3876=P(51)=P(P(6)$ and $C^4(17)=4845=P(57)=P(P(6)+6)$ – yes, I do want evidence!). Yet, I still wanted to *prove* that they are true. Then, having proved the three theorems, I felt successful … in a certain way; i.e., I now had mathematical certainty of the truth of all three statements. The proofs for the three were similar, fairly straightforward, but packed with many details, and they all used mathematical induction and within each induction argument a further mathematical induction argument. Thus, they were not *explanatory* to me! They didn't give me the insight I was looking for. The next section illustrates the kind of reasoning there is in the proofs in order to show why the proofs did not provide such insight.

Outline of the proof of Theorem 1. The proof uses mathematical induction and the following lemma, which itself is proved by mathematical induction: For any positive integers m and n, $\boldsymbol{P(m+n) = P(m) + 3mn + 2T(n\text{-}1) + T(n)}$, which I will refer to as the **critical lemma**. The proof of Theorem 1 starts by verifying that the theorem $C^4(3n\text{-}2)=\mathrm{P}(\mathrm{P}(n))$ is true for n=1, then assuming that it is true for an arbitrary integer n, and proceeding to show that, under this assumption, it is also true with n replaced by n+1; i.e., $C^4(3[n+1]\text{-}2)=P(P([n+1]))$. It then suffices to show that $C^4(3n+1)\text{-}C^4(3n\text{-}2)=P(P(n+1))\text{-}P(P(n))$. Now $P(n+1)=P(n)+3n+1$, so the right side of the equation can be replaced, using the critical lemma, by $3P(n)(3n+1)+2T(3n)+T(3n+1)$, and the left side, using the representation of C^4 as the sum of C^3 numbers, by $C^3(3n+1)+C^3(3n)+C^3(3n\text{-}1)$. From there, it is a direct, though messy, calculation using basic properties of C^4 in terms of C^3, of C^3 in terms of T, of $T(2n)$ and $T(3n)$ in terms of $T(n)$, and $P(n)$ in terms of $T(n)$ to get the proof. The proofs of Theorems 2 and 3 are very similar and also use the critical lemma.

Beyond proof. To get at "why," beyond proof, that the results are true, I tried visual representations of the two sides of the equations in the first two theorems and the three expressions in the third theorem, but I was unable to get a helpful visual representation of the Choose-4 expressions. So, I scaled back my goal to: Constructing a visual proof of the critical lemma, which is used in the proofs of all three theorems. An understanding of why the critical lemma is true can be gleaned from Figure 3.



$P(m+n)$ = P(m)+3mn+2T(n-1)+T(n) $\qquad$ $P(m+(n+1))$ = P(m)+3m(n+1))+2T(n)+T(n+1)

***Figure 3**: The Critical Lemma for m=5 and n=4*

Consider Figure 3: On the left, you can see how the 117 dots are partitioned into a pentagon of 35 dots, three quadrilaterals of (5)(4) dots each, two triangles of 6 dots each, and one triangle of 10 dots. I.e., $P(5+4) = P(5) + 3(5)(4) + 2T(4\text{-}1) + T(4)$. On the right, you can see how the 117+ 28 dots are partitioned into a again a pentagon of 35 dots, three quadrilaterals of, now, (5)(5) dots each, two triangles of 10 dots each, and one triangle of 15 dots; i.e., $P(5+4+1) = P(5) + 3(5)(4+1) + 2T((4+1)\text{-}1) + T(5)$. Together, one can see how the three parallelogram and three triangle structure is preserved when you add one more necklace of dots, thus going from $3mn + 2T(n\text{-}1) + T(n)$ to $3m(n+1) + 2T(n) + T(n+1)$. These pictures give me a deep and intuitive understanding of why the critical lemma is true.

Constructing the art. With this visual understanding of a key step in the proofs of all three theorems, I was inspired to construct a piece that would represent 70 as both a Choose-4 number and a pentagonal number and use a dot dissection similar to the one in Figure 3 that represents the critical lemma. I laid out the 70 dots in the standard figurate pentagonal way as illustrated in Figure 3. Instead of using the awkward partition of dots appearing in shaded regions, some on borders and some not, I constructed what

is called the Voronoi diagram for the 70 dots, so that each dot is "centered" in its own unique region. Being Voronoi means that all the points in a dot's region are closer to that dot than to any other dot. I 4-colored each Voronoi region from a palette of 8 colors: Red, bright green, blue, purple, brown, dark green, pink, and black, so that each of the 70 dots corresponds to a unique one of the 70 combinations of 8 colors chosen 4 at a time. The choice of the color combination for each region was made similar to the dot dissection in Figure 3, now with m=5 and n=2: P(7)=P(5)+3(5)(2)+2T(1)+T(2); i.e., 70=35+10+10+10+2+5. The coloring of regions corresponding, in order, to a. Pentagon with P(5)=35
b. Left parallelogram c. Right parallelogram d. Middle parallelogram e. Triangle with T(2)=3 and
f. Triangles with 2T(1)=2 is:

a. All 35 combinations containing color red
b. All 10 combinations containing both bright green and blue, but not red
c. All 10 combinations containing bright green, but neither red nor blue
d. All 10 combinations containing blue, but neither red nor bright green
e. All 3 combinations with both purple and brown, but neither red, nor bright green, nor blue
f. Both combinations containing dark green, pink, and black, and either purple or brown

Additionally, each region's 4-color combination preserves, in counterclockwise direction, the order red, bright green, blue, purple, brown, dark green, pink, and black. Finally, within the counterclockwise orientation of colors, the positions are chosen so that the entire 280-region pentagon is properly colored.

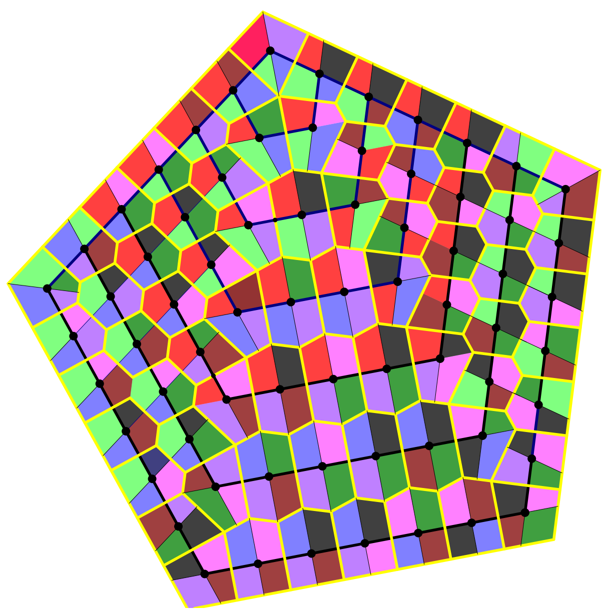


***Figure 4**: $C^4(5)= 70$ meets the $P(7) = 70$ in a 4-coloring pattern based on the Critical Lemma*

I achieved my goal of representing 70 in terms of a pentagon consisting of 70 dots in 70 polygons that are 4-colored using all 70 possible choose-4 combinations of 8 colors. I like the way that there is a pattern to the coloring that is inspired by the critical lemma, and I'm pleased to be doing mathematics inspired by art and art inspired by mathematics. Without the artistic endeavor, I would never have encountered the relationship of pentagonals to Choose-4's, which I found intriguing and which makes me wonder about where other pairs of sequences coincide. I'm happy also that there remains related mathematical art to do! For, it still isn't *intuitively* clear to me why exactly two of every three consecutive Choose-4 numbers must be pentagonal, while the third one always is strictly between two consecutive pentagonals. One of my goals for future work is to understand this relationship at a deeper and more intuitive level.