The Math and Art of Folded Books

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Abstract

We present aspects of book folding that are of mathematical interest including motivating problems and how the mathematics is useful both to analyze and to help generate the art. We also comment on the collaborative relationship between mathematician and artist.

Introduction

Book folding is the art of folding the individual pages in a book in such a way so as to produce an interesting work of art. Pages are usually only folded not cut. Multiple folds of the same page are allowed. Book folding is a restricted form of origami where one edge of each page remains fixed to the binding. Our interest is in how math can be used to both analyze the art and to aid the artist. Through two generic problems we'll consider the descriptive role math plays and then the potential for it to be prescriptive as well. Finally we opine our differing viewpoints.

Descriptive Role

In book folding, mathematics can play several roles ranging from descriptive to prescriptive as well as several intermediary roles which compromise neither the art nor the math. At first the use of math in our joint work was descriptive (and serendipitous). Sharol, the artist, had been folding books for a few years before Richard, the mathematician, took much notice.

On a sunny summer day after we had picked up one of Sharol's folded books that had been exhibited at the Minneapolis Institute of Art we went to visit our friend Roger Kirchner on Lake Minnetonka. He had just published a *Mathematica* Demonstration Project [1] about paper creasing, which he was only too glad to demonstrate. The three of us then confirmed that his paper creasing was exactly what was taking place in her retrieved folded book.

His project was about generalizations of the well-known greatest crease problem [2]: given a rectangular sheet of paper, what is the longest crease that can be formed when one corner of the sheet is folded to meet the edge on the opposite side.

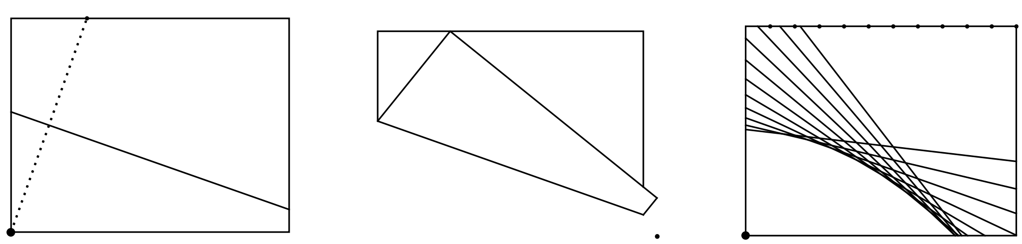

Figure 1: *Folding a corner to an edge*

The left panel in Figure 1 shows the crease that results when the lower left corner is folded to the point indicated on the upper edge. Note that the distances to the crease are equal from either end point along the dotted perpendicular line. The middle panel shows what the folded sheet would look like. The right panel shows the creases resulting from folding the corner to eleven successive points along the upper edge. The nearly horizontal crease results from folding to the point one unit from the upper left corner and the steepest crease results from folding to the upper right corner. Note that there are three cases for the creases shown. They can connect the left and right edges, the left edge to the bottom, or the top to the bottom. If the paper's width was less than the height a top-to-the-right-edge crease would also be possible. This greatest crease problem is valued because a thorough analysis requires that the domain of interest be subdivided into several subdomains any one of which might yield the answer depending on the relative sizes of the edges.

For a simple folded book the top edge of the sheet (page) will be identified with the binding and successive pages folded to a succession of points on the top. However, after a few folds we would have to stop to avoid a crease end lying on the binding. For the actual folded books Sharol continually modified the choice of folding to avoid such problems as well as to see where the work was taking her. Later, Richard analyzed what alternative steps were taken by considering where the folds were located.

Prescriptive Role

Ideally (rather naively) the next stage was for Richard to dictate to Sharol exactly how to fold each page: which corner to fold, to what point it should be folded, whether there should be a subsequent folding of the same page, what the increments should be, how many pages to fold, how far the book should be opened and whether it should lie flat or stand upright. That didactic approach didn't work well. Sharol ignored most suggestions and continued to fold books intuitively. To see what did work, let's first consider an alternative, although equivalent, approach to folding.

Rather than folding a point to an edge, an edge can be folded to a point. This is the classical case for producing a parabola [2]. Consider the left panel in Figure 2 in which the bottom edge is folded along the crease (solid line) so that point A goes to the focus point. The right panel shows a sequence of such folds, all folding the lower edge to the focus. The curve taking place seems to be parabolic. Indeed the creases are the tangent lines, or envelope, for a parabola. To see this more clearly, reconsider the left panel. Note that the perpendicular distance from the point P to the bottom edge equals that from P to the focus. Thus all such points P satisfy the classical definition of a parabola being the locus of all points equidistant from a point (the focus) and a line (the directrix), which here is the bottom edge.

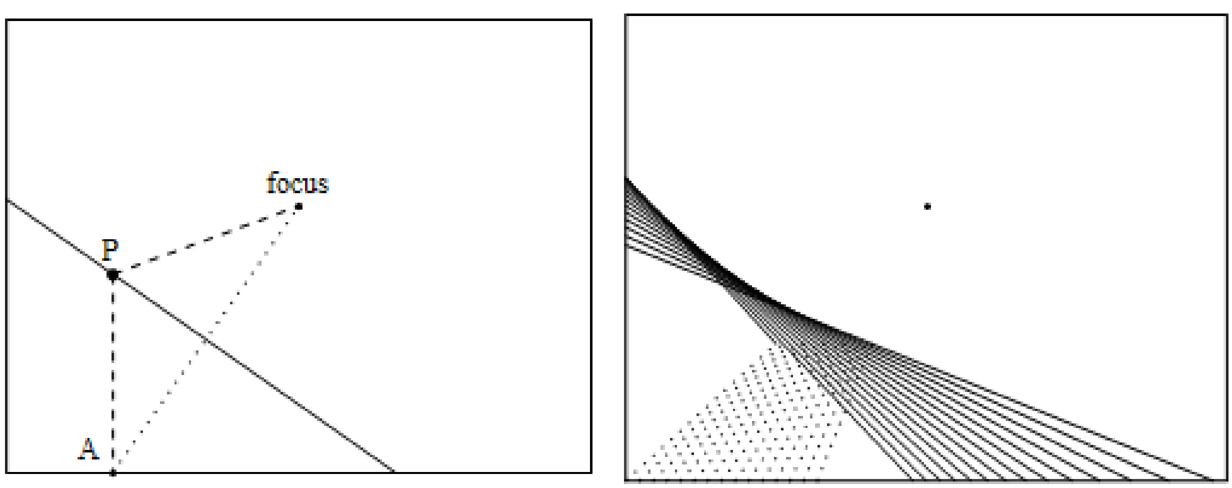

Figure 2: *Folding an edge to a point*

Using rectangular coordinates with the origin at the center of the sheet, for a sheet of height H, width W and focus (0,c), the equation of the parabola is simply $x^2 = 4\,c\,y$.

We now also have knowledge of what the curve formed in the greatest crease problem is. The creases there form an envelope to a parabola, whose focus is the lower left corner. It must be a parabola because folding an edge to a point is equivalent to folding a point to an edge. Taking the origin to be at the lower left hand corner of the sheet, the equation for the parabola is $y = -\,x^2/\,(2H) + H/2$.

For the folds in Figure 2, let each sheet, taken in order, be a page bound on the upper edge in a book. Then setting the assemblage upright we arrive at the model in the left panel of Figure 3. This modeled book can be manipulated in various ways including rotating it into any position, modifying the width and height of the pages, and setting the spacing between the folds, the number of pages, how far the book should be opened, etc. The right panel of Figure 3 shows a similarly produced folded book, *The Great Imposter*. The titles used here for the actual folded books are identical to those of the unfolded books.

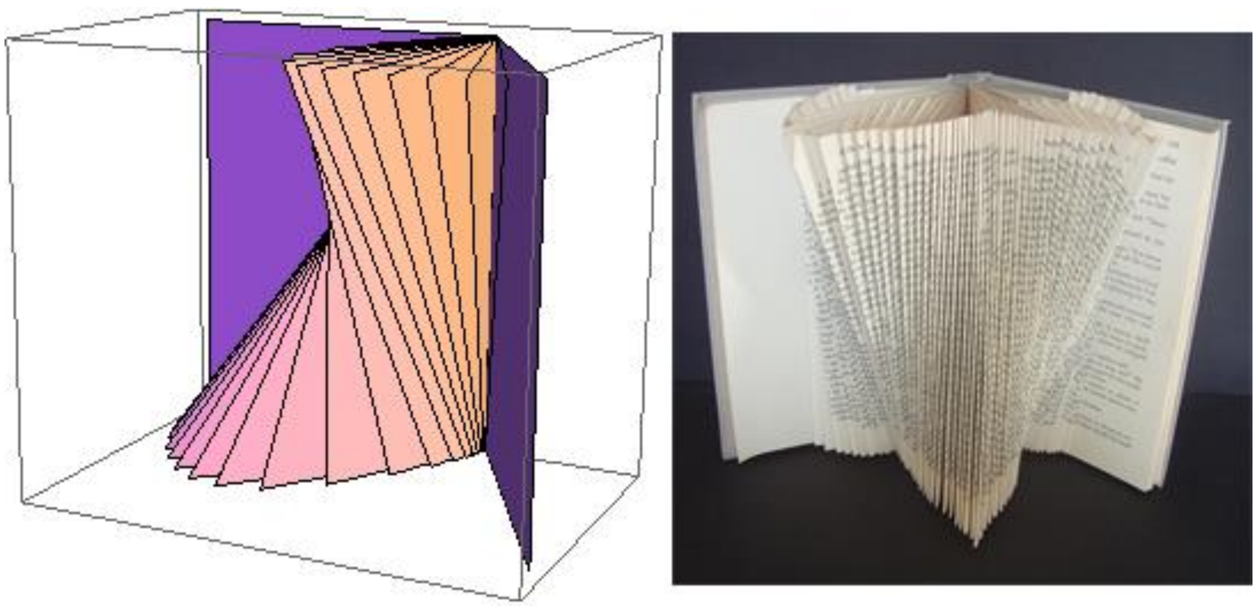

Figure 3: *A Model and The Great Imposter*

Summaries

Richard's Viewpoint. We now have two types of tools that the artist can use effectively. Time saving templates similar to the right panels in Figures 1 and 2 can be used to help transfer points and tangent and normal lines from one page to the next. Although not providing a definitive goal for the artist, the manipulated model is a useful mock-up to provide additional variations to be considered. Our fear that somehow this close probing into her work would adversely affect her creativity was unrealized. The opposite was true as it did provide new directions and exposure. Furthermore, dynamical presentations of the manipulated computer graphics models can often be revealing and artful in their own right..

Sharol's Perspective. Mathematics came into play only after I had folded several volumes from a set of encyclopedias for the practice and fun. Being material conscious, I switched to novels which I can purchase by the bagful at an annual book recycling sale. There are still many excellent hardcover books to choose from on the last day of the sale which otherwise would likely go into a shredder or landfill. I prefer novels that have been printed using better paper, although not necessarily of archival qualty, especially those with deckled edges.

Mathematics has enhanced the variety of the three-dimensional forms obtainable. The conscious choice of parameters for the folds and the parabolas allows for a diverse exploration in the design. The wave-like forms resulting from the sequence of folds are varied and beautiful. See Figure 4.

These works have appeared on the cover of journals, as the picture for October in the 2016 AMS's calendar, in a two-year traveling tour of book art at libraries, on the web [3]. at galleries and in collections in several states and now in Finland.

Conclusion

Art is not just the lines and points we see on a piece of paper. It's what our mind's eye beholds between the lines. The mathematician in us views surfaces, shapes, curvatures and perhaps the underlying equations. The artist in us may also perceive the crying of a dove, the heady perfume and sweet innocence of a lily-of-the-valley on a spring day, or the warmth of a sauna on a cold winter's night.

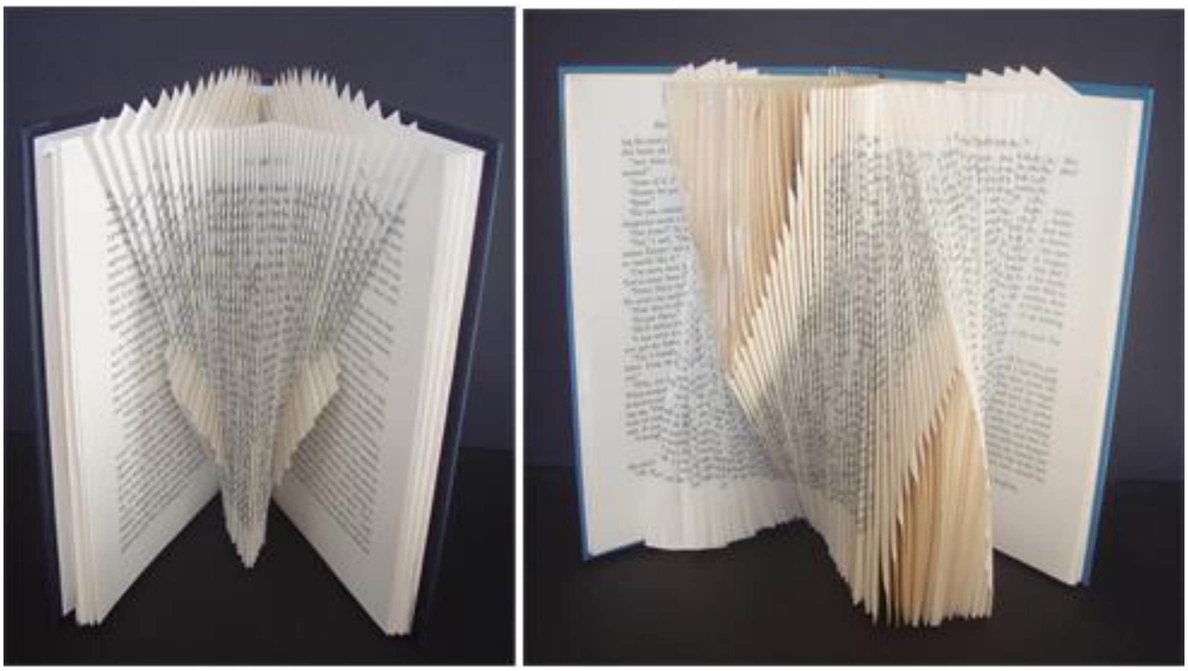

Figure 4: *Holding My Breath and Castle Ugly*

References

[1] Roger B. Kirchner, Generalizing the Crease Length Problem, Wolfram Demonstrations, http://demonstrations.wolfram.com/GeneralizingTheCreaseLengthProblem, (as of Feb 29, 2016)

[2] Martin Gardner, *Second Scientific American Book of Mathematical Puzzles and Diversions,* Univ. of Chicago Press, 1987, p. 177–8

[3] Sharol Nau, Mathematical Art Galleries, (May 15, 2013) gallery.bridgesmathart.org/exhibitions/2012-joint-mathematics-meetings/sharol-nau. (as of Feb 29, 2016)