Euler-Cayley Formula for ‘Unusual’ Polyhedra

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Abstract

Euler’s formula for a polyhedron with V vertices, F faces and E edges and genus g states that $V + F - E = 2(1 - g)$. Cayley formulated a similar expression for the four Kepler-Poinsot polyhedra: $bV + aF - E = 2c$, using the densities a, b and c of respectively the faces, the vertices and the polyhedron itself. We conjecture both should be united as $bV + aF - E = 2c(1 - g)$, or, more generally, for Archimedean polyhedra with V_j vertices and F_i faces of a given type and densities b_j and a_i respectively, as $\Sigma b_j V_j + \Sigma a_i F_i - E = 2c(1 - g)$. We consider examples that support our case, but a more important aspect of this paper is perhaps the suggestion of considering higher genus Kepler-Poinsot polyhedra. Some are unusual cases inspired by S. Gott’s pseudo-polyhedra with adjacent faces in the same plane, while other types of these star polyhedra may be attractive from an artistic point of view. Escher was interested in star polyhedra and in infinite tessellations, and the polyhedra proposed in the present paper combine both topics.

Introduction

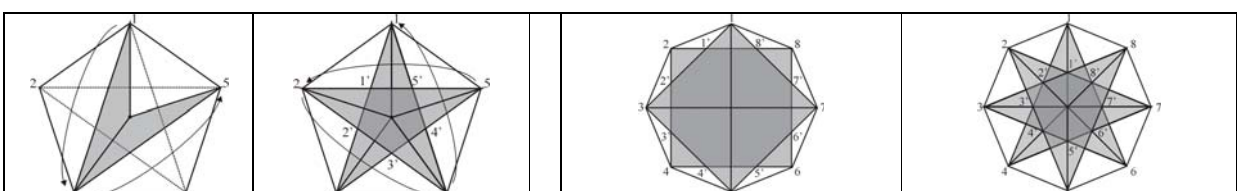
Euler’s formula states that for a convex polyhedron with V vertices, F faces and E edges:

$$V + F - E = 2.$$

It is also known as the *Descartes-Euler polyhedral formula*, since Descartes discovered it independently. The expression $V + F - E$ is known as Euler’s characteristic and denoted by χ. Cayley generalized Euler’s formula to regular star polyhedra, using the face density a, the vertex figure density b, and the polyhedron density c:

$$bV + aF - E = 2c.$$

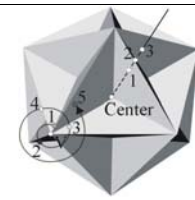
The face density a of a star polygon is 2 if one vertex is omitted to reach the next vertex of the subscribed polygon in a given star polygon (as in a star pentagon, see Figures 1a and 1b), 3 if two vertices are omitted (thus, for a star heptagon or octagon, a can be 2 or 3; see Figures 1c and 1d), and so on. It can also be seen as the number of times the star polygon goes around the center. Therefore, some call it the ‘winding number’, that is, the number of times a string surrounds a finger placed at the center.



a. A pentagram in a pentagon by omitting one vertex, first steps.	*b. The middle pentagon is covered twice and has face density a = 2.*	*c. An octagonal star omitting 1 vertex; the middle octagon 1’2’…. is twice as grey.*	*d. An octagonal star omitting 2 vertices; for the middle octagon 1’2’…: a = 3.*

Figure 1: *Some examples explaining the face density.*

Similarly, the vertex figure density *b* tells us how many times a vertex should be counted when forming a polyhedron. It is the 'winding number of the vertex figure', the polygon that results when the vertex is slightly truncated. The polyhedron density *c* indicates by how many layers the polyhedron would cover a sphere if the polyhedron were blown up to a sphere (see Figures 2b, c and d; [8] provides a discussion about these definitions). In many cases *c* (but not all) can be interpreted as the number of intersections the polyhedron has with a half-line originating at its center, taking into account the density of each face it passes through. For regular convex polyhedra all density values equal 1, and thus Cayley's formula is a generalization of Euler's formula (see Figure 2d).

<table>
<tr><td>a. Dodecahedron</td><td>b. Small dodecahedron</td><td>c. Great dodecahedron</td><td rowspan="4">

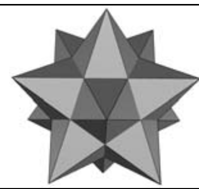
b = 2: one goes around the vertex V twice; there are c = 3 intersections</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>$V = 20, F = 12, E = 30$</td><td>$V = 12, F = 12, E = 30$</td><td>$V = 12, F = 12, E = 30$</td></tr>
<tr><td>$20 + 12 – 30 = 2$</td><td>$1×12 + 2×12 – 30 = 2×3$</td><td>$2×12 + 1×12 – 30 = 2×3$</td></tr>
</table>

Figure 2: *Classical cases and an illustration of the vertex density b and polyhedron density c.*

Cayley's generalization made Euler's formula applicable to the Kepler-Poinsot polyhedra. His formula also works for star polyhedra 'of Archimedean type', that is, for star polyhedra that use more than one type of regular polygon for the faces (of course, there are but 13 or 14 Archimedean solids, but by extension we use this indication here too). Suppose it has V_j vertices and F_i faces of a given type, with densities b_j and a_i respectively. Then following the generalization of Cayley's formula, we obtain:

$$\Sigma b_j V_j + \Sigma a_i F_i – E = 2c.$$

An example of Archimedean type is the truncated great icosahedron, formed by 12 pentagrams and 20 hexagons. If it were blown up to a sphere, it would cover the sphere 7 times, and thus $c = 7$. The 7 points of intersection of a half-line from the center are shown here but it would bring us too far to explain where these intersections come from (see Figure 3a). Fortunately, there are references explaining this in more detail [8]. The formula also works for compound polyhedra such as the Kepler star, when it is seen as two joint tetrahedra. The formula 4 + 4 – 6 = 2 is simply doubled and becomes 8 + 8 – 12 = 2×2. Here *c* is obviously 2 since there are 2 layers, one from each tetrahedron composing the Kepler star. This case is perhaps not very relevant, sure, but it does provide a nice illustration of the main ideas (see Figure 3b). A third example uses an open pentagonal anti-prism (an anti-prism with triangular faces with pentagrams on the bottom and the top) on which stand star pyramids on the top and on the bottom. The top and bottom vertex have a density 2. This polyhedron is new, perhaps. Again, *c* is 2, as can be understood straightaway by comparison with the Kepler star (see Figure 3c).

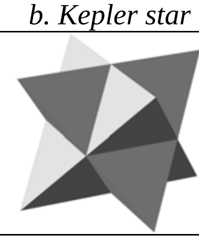
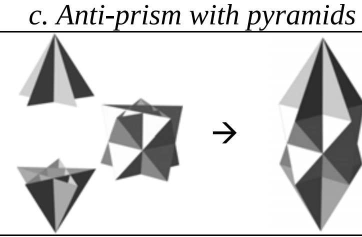
a. Truncated great icosahedron	*b. Kepler star*	*c. Anti-prism with pyramids*
		
$V = 60, F_1 = 12, F_2 = 20, E = 90$	$V = 8, F = 8, E = 12$	$V_1 = 10, V_2 = 2, F = 20, E = 30$
$60 + (2×12 +20) – 90 = 2×7$	$8 + 8 – 12 = 2×2$	$(10 + 2×2) + 20 – 30 = 2×2$

Figure 3: *A case 'of Archimedean type', and some less usual cases.*

For higher genus polyhedra there is another generalization of Euler's formula. The genus g is the maximum number of cuttings along non-intersecting closed simple curves without disconnecting it, or else, the number of loops, handles or holes. For an infinite polyhedron, the definition of the genus is applied to one of the repeating parts (please see the numerous references for a definition). Now the formula becomes:

$$V + F - E = 2(1 - g).$$

For instance, for a toroid of 8 octahedra, $V + F - E = 0$ and $g = 1$. Each of the 8 'atoms' is formed by an octahedron of which 3 vertices, 2 faces and 3 edges are removed, as the removed parts fit on those of the atoms next to it (see Figures 4a and 4b). For Coxeter's infinite polyhedron {6, 4} four hexagons meet at a vertex and $V + F - E = -4$ as $g = 3$. Each 'atom' is formed by an octahedron of which 3 times 4 vertices, 6 faces and 3 times 4 edges are removed. Again, the removed parts fit on those of the atoms next to it (see Figures 4c and 4d; the removed vertices and edges were represented in black).

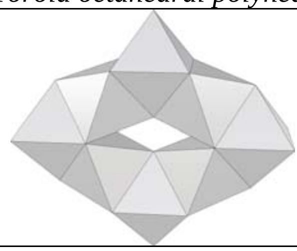
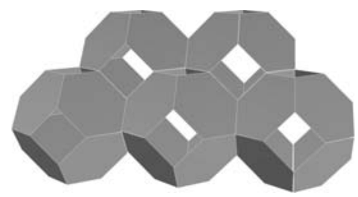
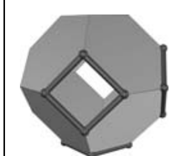
<table>
<tr><td colspan="2">a. Toroid octahedral polyhedron and one 'atom'</td><td colspan="2">b. Coxeter's {6, 4} and one 'atom'</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>$V = 8×3$, $F = 8×6$, $E = 8×9$, g = 1</td><td>$V = 3$, $F = 6$, $E = 9$</td><td colspan="2">$V = 12$, $F = 8$, $E = 24$, g = 3</td></tr>
<tr><td>8×3 + 8×6 – 8×9 = 2× (1 – 1) = 0</td><td>3 + 6 – 9 = 0</td><td colspan="2">12 + 8 – 24 = 2× (1 – 3) = –4</td></tr>
</table>

Figure 4: *Illustrations of Euler's formula for the non-zero genus case.*

Joining Euler's Genus Formula and Cayley's Formula

Uniting the above formulas suggests an Euler-Cayley genus formula for Archimedean polyhedra with V_j vertices of densities b_j and F_i faces of a given type and densities a_i:

$$\Sigma b_j V_j + \Sigma a_i F_i - E = 2c(1 - g),$$

where c is the number of layers of each composing element or 'atom' of the (infinite) polyhedron as seen from its center. This formula seems just a multi-layer version of Euler's polyhedron formula for arbitrary genus combined with Cayley's formula for Kepler-Poinsot solids. We now provide examples that seem to confirm the validity of this formula.

An example like the above toroidal octagon ring is a ring of 22 *small ditrigonal icosidodecahedra* (see Figure 5a). For one small ditrigonal icosidodecahedra, $V = 20$, $F = 32$ (20 triangles and 12 pentagonal stars) and $E = 60$. To form an atom of a ring, 2 pentagonal faces are removed, including, for one of them, the vertices and the edges, so that $V = 15$, $F = 30$ and $E = 55$ (see Figure 5b). Where 2 rings meet, 3 pentagonal faces have to be removed. They can be placed in 3 rings (or more), thus forming a genus 3 example. An example of an infinite polyhedron is a column of open pentagrammic antiprisms of which the pentagrams have been removed. In this case, 6 triangles meet at each vertex so that it is an example of a layer 2 case. As the genus is 1, the right-hand side of the higher genus Euler-Cayley formula, $2c(1–g)$, is 0 (see Figure 5c).

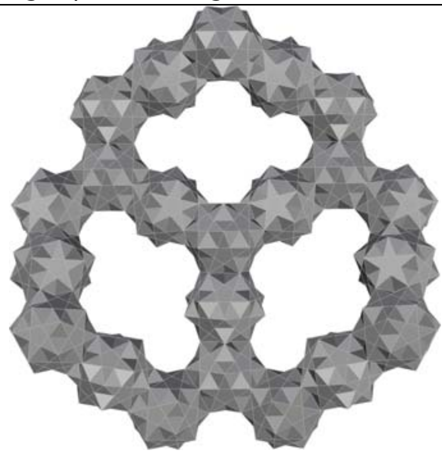
a. Rings of small ditrigonal icosidodecahedra	*b. Atoms*	*c. Tower of star antiprisms*
		
V = 200, F_1 = 440, F_2 = 216, E = 1080		$V = 5$, $F = 10$, $E = 15$
200 + 440 + 2.216 – 1080 = –8 = 2.2.(1–3)		1×5 + 1×10 – 15 = 0 = 2×2×(1–1)

Figure 5: *Examples for the Euler-Cayley formula for higher genus.*

More 'Unusual' Examples

Coxeter excluded shapes with adjacent faces in a common plane to be considered as polyhedra (see [2]). However, this rule hasn't been followed by all authors, such as J. R. Gott, who called some of the solids he discovered 'pseudo-polyhedra' (see [5]). Others did not make that distinction and simply called them 'polyhedra'. This seems quite a topic of discussion and so we refer to several references, such as [1], [4] or [9].

Nevertheless, a perhaps new and surely 'unusual' (pseudo)-polyhedron is a *closed toroid star polyhedron* where 6 equilateral triangles meet in each vertex. All triangles are equilateral but some cut each other causing an apparent trapezoid intersection. Three triangles lay in one plane along an edge of a pentagram and perhaps this explains why this example is has not been mentioned in literature before; it has $g = 1$ and $c = 1$ (see Figure 6a). An example of a higher c case is formed by star prisms, where $g = 1$ but $c = 3$ (see Figure 6b). Another example of an infinite polyhedron of genus 2 with $c = 2$ can be made based on an example by Gott. He proposed a genus 2 infinite polyhedron in which 8 triangles meet in each vertex. A 60° rotation yields a hexagrammic star and thus a layer 2 infinite (pseudo-) polyhedron of Kepler-Poinsot type (see Figure 7).

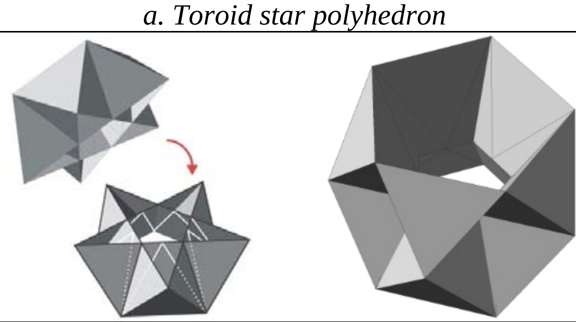
a. Toroid star polyhedron	*b. Octagrammic star prisms*
	
$V = 20$, $F = 40$, $E = 60$	$V = 4$, $F = 4$, $E = 8$
20 + 40 – 60 = 2×(1–1)	1×4 + 1×4 – 8 = 2×3 – 2×3

Figure 6: *A pentagrammic ring.*

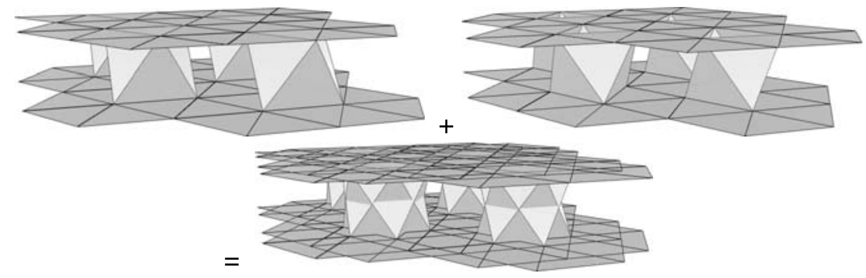
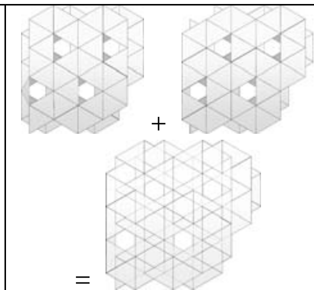
	
$V = 12$, $F = 32$, $E = 48$ and so: $12 + 32 - 48 = 2\times2\times(1 - 2)$	*Corresponding top views*

Figure 7: *Combining 2 copies of Gott's {8,3} infinite (pseudo)-polyhedron.*

The previous examples confirmed the conjectured generalization of the Euler-Cayley formula $bV + aF - E = 2c(1 - g)$. The faces of the atoms were regular *n-gons* and so when one face with its vertices and edges was deleted from the polyhedron, the formula $bV + aF - E = 2c$ became:

$$b(V - n) + a(F - 1) - (E - n) = 2c - a + n(b - 1)$$

or

$$V' + aF' - E' = 2c - a + n(b - 1),$$

where V', F' and E' are the values of the corresponding polyhedron minus that one face. If p faces are deleted the right hand side becomes $2c - p[a - n(b - 1)]$. Thus, in case $b = 1$, deleting p faces, with or without their vertices and edges, diminishes the right hand side by pa. The numbers p, a, c and g often are small natural numbers and so it is not so surprising that $2c - pa$ would equal $2c - 2cg$. Thus, it is perhaps sheer coincidence that the conjectured formula was validated in the previous examples.

Here is an example where these numbers are slightly larger: consider *open great rhombihexahedrons* connected by *octagrammic tunnels* (see Figure 8c). A 'regular' (closed) great rhombihexahedron has 24 vertices, 6 octagrammic stars and 12 square faces, and 48 edges, while an octagrammic prism has 16 vertices, 2 octagrammic stars and 8 square faces, and 24 edges (see Figure 8a). Three of the latter can be 'glued' on one great rhombihexahedron, omitting the octagrammic stars. Together, they will have 24 vertices, 36 faces and 72 edges if the commons vertices, faces and edges are not counted (see Figure 8b and 8c).

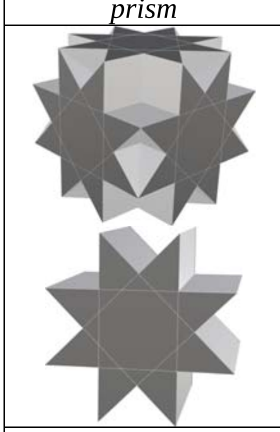
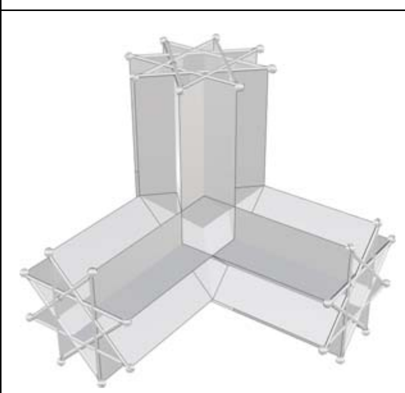
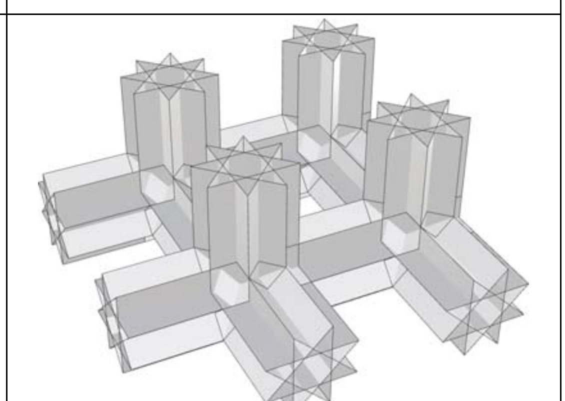
a. A great rhombihexahedra and an octagrammic prism	*b. Atom: 3 octagrammic prisms 'glued' on an open great rhombihexahedron.*	*c. Four atoms combining to an 'infinite Kepler-Poinsot-polyhedron'.*
		
$V_1 = 24$ $F_1 = 20$, $E_1 = 48$ and $V_2 = 16$, $F_2 = 10$, $E_2 = 24$	*Here, 24 vertices, 36 faces and 72 edges are left.*	*For the combination:* $V = 24$, $F = 36$, $E = 72$, *and* $24 + 36 - 72 = -12 = 2\times3\times(1-3)$

Figure 8: *An infinite Kepler-Poinsot polyhedron?*

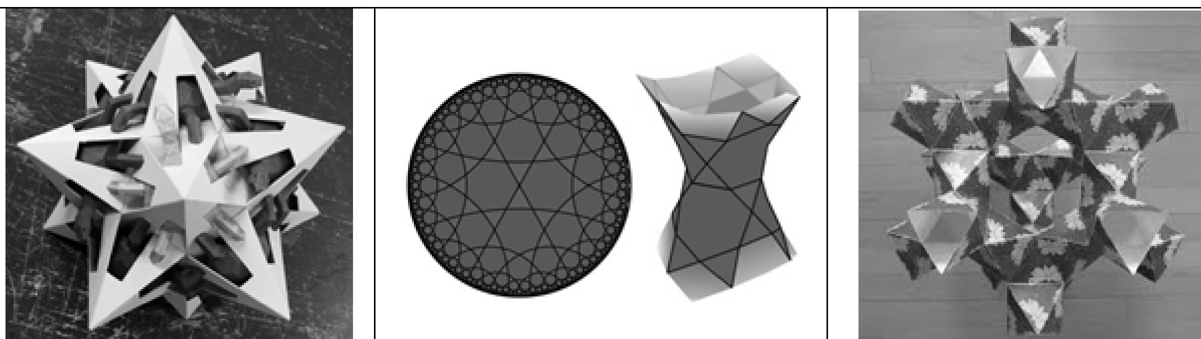
Thus, 6 squares meet at each vertex:

$$V + F - E = 24 + (12 + 3\times8) - (48 + 3\times8) = -12 = 2\times3\times(1 - 3) = 2\times c\times(1 - g).$$

However, the atoms are not connected by ‘central tunnels’ so that one cannot go from one part to another inside the infinite (pseudo)-polyhedron, as in the known infinite polyhedra. For instance, in the given infinite prisms there is a central tunnel, while Coxeter’s {6, 4} example has two ‘horizontal’ and one ‘vertical’ tunnel. And still, perhaps the example given here in Figure 8 could be seen as infinite Kepler-Poinsot polyhedron.

Artistic Considerations

The infinite Kepler-Poinsot polyhedra could inspire artists. Escher for instance illustrated the many layer property of the small stellated dodecahedron by letting little tortoises crawl in and out of the polyhedron (see Figure 9a; we used the ©free image given in [10]). In a 2012 paper I suggested representing hyperbolic tessellations on a hyperboloid (see Figure 9b and [6], [7]). Douglas Dunham represented Escher’s hyperbolic tessellations on infinite polyhedra (see Figure 9c and [3]). And so, perhaps future artists will make infinite Kepler-Poinsot polyhedra more acceptable to our intuition by proposing infinitely many crawling tortoises on infinite star structures.



a. An Escher type star polyhedron artwork.	*b. A {7,3,7,3} tessellation by R. Sazdanovic and on a hyperboloid.*	*c. Dunham’s Escher butterflies on an infinite polyhedron.*

Figure 9: *Inviting artists to illustrate infinite Kepler-Poinsot polyhedra.*

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