Theory of Intersection

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Abstract

While contemplating the flow of traffic at a roundabout, a combinatorial problem arises. Its unique solution is found and diagrammed. By tiling the plane with copies of this diagram, we discover a nonconvex polygon, and this polygon is depicted in the abstract painting *Theory of Intersection* (Lisa Kattchee, Karl Kattchee, 2014).

Introduction

The purpose of this paper is to derive the polygonal form on which the painting *Theory of Intersection* (Kattchee, Kattchee, 2014) is based. See Figure 12.

The project started with explorations in automatic drawing. Many of my doodles were closed paths that swirled around a common center, as in Figure 1**.** I wondered how such paths might be classified.

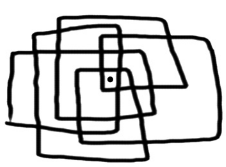

Figure 1

Figure 2

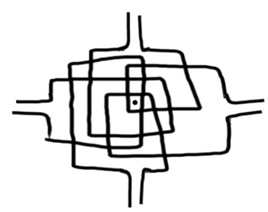

Figure 3

At Bridges Enschede (2013), Craig Kaplan presented his paper *Grid-Based Decorative Corners* [1], in which the idea of "(n,k)-corner" is introduced. Briefly, the idea is that one set of k parallel lines heads across the bottom of a page, and another set is heading down the right side. They interact in the corner of the page, within an $n\times n$ grid of squares. If the interaction satisfies certain conditions (including symmetry across the main diagonal of the grid), then the result is called an "(n,k)-corner design." Aesthetic restraints were imposed as well. Kaplan shows how enumeration of these (n,k)-corners is possible.

Figure 2 shows an example of a (6,*2*)-corner [1, p.319]. Kaplan refers to the two parallel lines on the lower left as "input wires." The "output wires" are the two parallel lines on the upper right. By a simple modification of my automatic drawing, we get Figure 3, which has eight "wires." Four of the wires are inputs, and four of them are outputs, according to the way traffic flows in and out of a typical intersection. In the next section, we will describe how our wires interact.

The Roundabout Problem

The path of traffic at an intersection is depicted in Figure 4**.** The vertices v_1, v_2, v_3, and v_4 indicate points where a vehicle might change course, with the proviso that traffic must proceed counterclockwise around the central disk, as in a roundabout or traffic circle.

Because we are viewing the intersection as a roundabout, it is conceivable that a vehicle's path might circulate multiple times around the central disk in the middle of Figure 4 before departing the intersection.

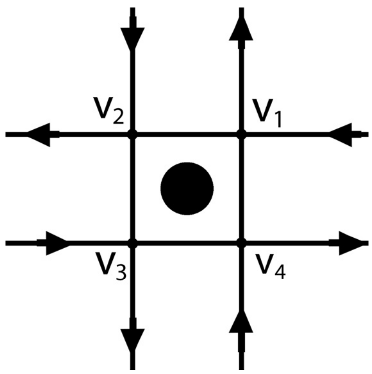

Figure 4

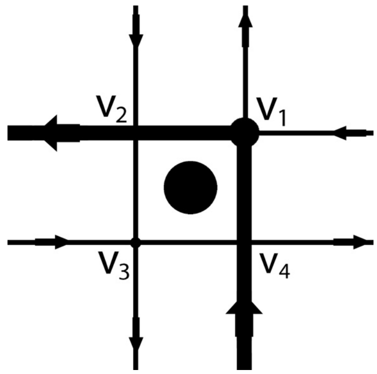

Figure 5

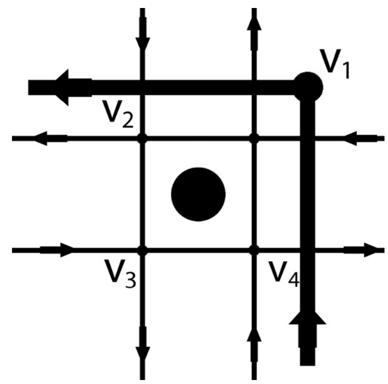

Figure 6

If the paths of the four vehicles navigating the intersection are traced, we could not recover the routes taken by the individual vehicles unless we had encoded each path with a unique color. Without such a coloring scheme, Figure 4 would not indicate whether a given vehicle proceeded straight through the intersection without changing direction, or if it drove all the way around and exited the roundabout in the opposite direction. Or if it made a left turn. Or a right turn. We couldn't be sure just by looking at the diagram.

For convenience, let us agree that all vehicles are making *left turns* at the intersection. Each path is divided into two perpendicular "half-paths" by a vertex (see Figure 5). Other patterns are certainly of interest, but we will deal with only this one.

Even though we have agreed on this simple flow of traffic, nothing has been done to deal with the ambiguity of Figure 4 in depicting the precise paths. We would like to render this traffic pattern in such a way that the individual paths are distinguishable. To do this, we allow ourselves the flexibility to relocate each vehicle's path. Figure 6 shows how one path, along with its vertex, can be translated to the right and upwards so its entirety is discernible. By doing this sort of thing to each path, we can distinguish them all from each other. See Figure 7**,** for example. We'll refer to such a diagram as "unambiguous." The unambiguous diagrams are characterized by the fact that no pair of half-paths are collinear.

The task of constructing an unambiguous diagram boils down to a choice between four possible positions for each vertex so that no two half-paths are collinear. For example, whether or not vertex v_1 should be moved to the right depends on the location of vertex v_4, and whether or not it should be moved up depends on the location of vertex v_2. In Figure 8, for example, vertex v_1 has been moved right and up, while v_4 has not been moved at all.

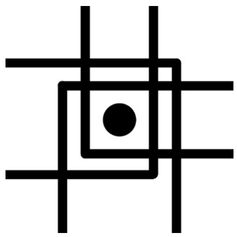

Figure 7:

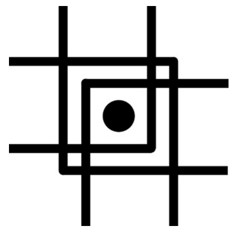

Figure 8:

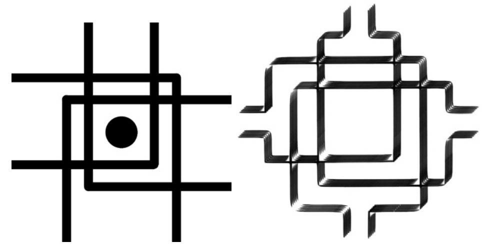

Figure 9:

One can easily check that, under the setup above, there are three distinct unambiguous diagrams (up to symmetry), and they are pictured in Figures 7-9 above. Evidently, there is only one way to construct an asymmetrical diagram. It is shown in Figure 9, along with a stylized version for use in the next stage of the process.

Polygons

Figure 10 shows a detail from an Islamic interlace pattern. Figure 11 shows the result of tiling the plane with copies of the unambiguous diagram in Figure 9. In the same way an eight-pointed star arises from the interlace pattern, a nonconvex polygon arises from Figure 11. This polygon is the subject of the painting *Theory of Intersection* (Lisa Kattchee, Karl Kattchee, 2014). See Figure 12 below.

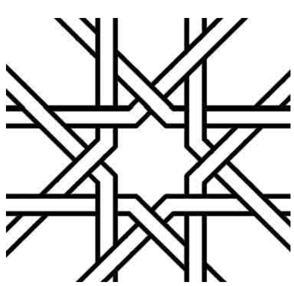

Figure 10

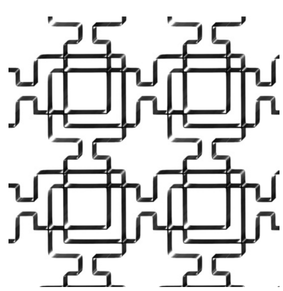

Figure 11

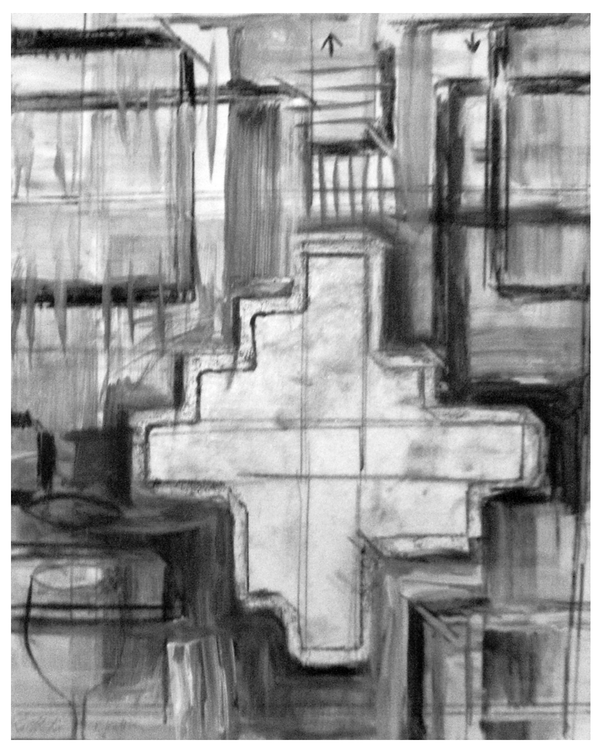

Figure 12: *Theory of Intersection*

To take the process one step further, let us divide our nonconvex polygon into quadrants, as in Figure 13. Note that I and IV are identical. There are three distinct nonconvex polygons that can be formed by rearranging the four quadrants (no flipping). They are all asymmetrical, and they are pictured in Figures 13 and 14.

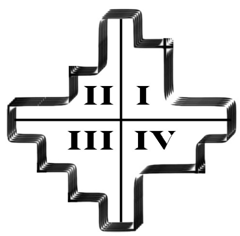

Figure 13: *Polygon* A

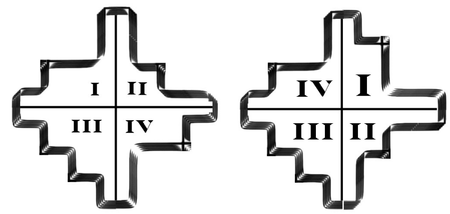

Figure 14: *Polygons* B *and* C

Recall that Polygon A in Figure 13 arose naturally after tiling the plane with an unambiguous intersection diagram. Can the polygons in Figure 14 be similarly derived? The answer is yes, and Figure 15 shows how Polygon B can be derived. The case of polygon C is left as an exercise. Note that, in the diagram which gives rise to Polygon B, only one of the four input wires leads to a left turn. Two of the other paths proceed straight through, and the fourth makes a right turn, but only after circling the roundabout once.

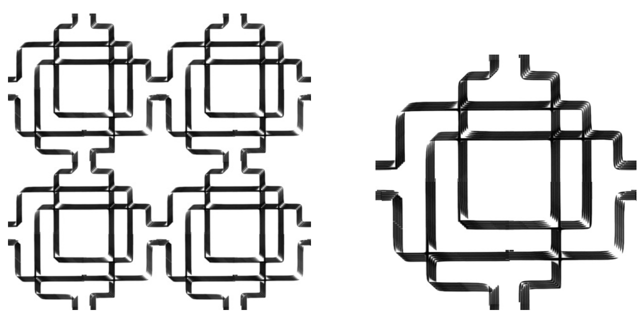

Figure 15

Conclusion

In this paper, we derived the polygonal form which appears in *Theory of Intersection*. The derivation was a mathematical process, simplified greatly by imposing certain restrictions (e.g. left turns only). In future work, it might be interesting to classify the variety of diagrams and polygons which arise in more general settings.

I am grateful to the referees for their many helpful suggestions.

References

[1] Craig S. Kaplan. *Grid-based decorative corners***,** Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture (2013) Pages 317–324