Katzengold: Pyrite, Plato, and a Polynomial

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Abstract

What are the similarities of the mineral pyrite, the dodecahedron as the fifth Platonic solid, and a polynomial of degree 16? This paper explores this connection by using the free software SURFER of the IMAGINARY open mathematics platform, which leads to fascinating pictures displaying transformations from a cube to a dodecahedron, to a rhombic dodecahedron, and to an octahedron, using a single formula. A survey on the ideas and the mathematics behind these visualizations is given. In fact, everyone can explore and modify these forms in real time using the SURFER software. Moreover, the authors have created a short movie, demonstrating the simple geometric beauty.

Chemistry and Ancient Greek Scientists

Katzengold is the German name for *fool's gold*, the common nickname the mineral pyrite has received because of its superficial and illusory resemblance to gold. Pyrite is an iron sulfide with the chemical formula FeS_2 and is the most common one of the sulfide minerals [1]. It usually forms cuboid crystals (Figure 1.a), but it can also form dodecahedral crystals (Figure 1.b), which is a rare property among minerals. Note that the dodecahedral structure is composed of two different edge lengths and is therefore irregular. Crystals cannot be formed by absolutely regular dodecahedra [2].

(a) cuboid

(b) dodecahedral

Figure 1 : *Crystals of pyrite* [1]

We make a short digression into the five Platonic solids, which are the tetrahedron, the hexahedron (or cube), the octahedron, the dodecahedron, and the icosahedron (Figure 2). A Platonic solid is a convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex. It is a mathematical truth that there are exactly five Platonic solids [3] and their Greek names refer to their number of faces (tetra $= 4$, hexa $= 6$, octa $= 8$, dodeca $= 12$, icosa $= 20$). Greek geometers and philosophers, in particular Plato, have studied their mathematical beauty and symmetry for thousands of years. Moreover, four of them are attributed to the four classical elements of alchemy: fire = tetrahedron, air = octahedron, water = icosahedron, and earth = cube. Aristotle added the aether as a fifth element, from which heavenly,

Figure 2 : *The five Platonic Solids* [4]

supernatural, and translucent things like the spirit and celestial spheres supporting the stars are formed. The symbol of this *quintessence* is the dodecahedron [4] which is the only Platonic solid formed by pentagons.

Algebraic Geometry

Now let us turn to *algebraic geometry*, more precisely to real algebraic geometry in 3 dimensions. Suppose we are given a single polynomial equation $p(x, y, z) = 0$ in three variables x, y, and z, then a solution of this equation consists of specifying three values for the variables which can be interpreted as a point in 3-dimensional space specified by its coordinates [5]. Mathematical reasoning shows that, in general, the set of *all* solutions forms a curved surface, possibly with singularities like self-intersections or cusps. Thus an algebraic equation $p(x, y, z) = 0$ produces a surface as a geometric object, i.e., a formula creates a form [6]. Very often it is possible to create astonishing and aesthetic surfaces by relatively simple equations [7], and there are deep connections between formulas and forms, which are only partially understood by modern mathematical research.

Here we should emphasize that it is possible to visualize the algebraic surface of a given polynomial equation in real time using the free software SURFER [8] of the IMAGINARY open mathematics platform [9]. This software is quickly installed and can be used very easily also by non-mathematicians [5]; it was developed for the general public during the German year of mathematics in 2008. Let us start with the cube, which can be approximated by the simple equation $x^{16} + y^{16} + z^{16} = 1$. Note that this is a 3-dimensional generalization of the Lamé curve [10] which was also used in the work of Piet Hein, the famous Danish scientist, artist and designer. Figure 3 shows a picture generated by the authors using the SURFER software:

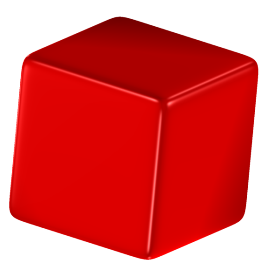
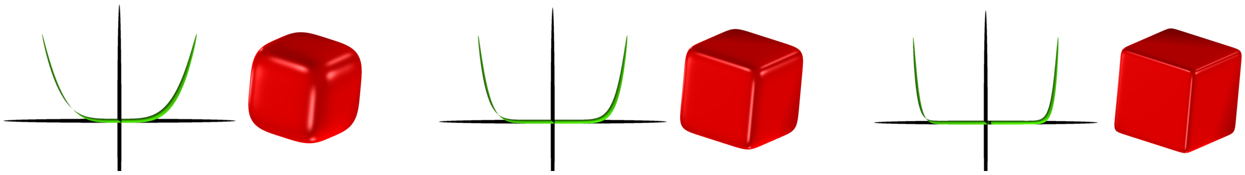


Figure 3 : *Approximation of a cube*



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$

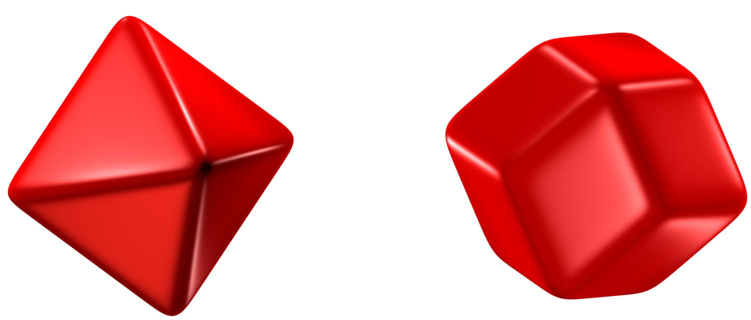
Figure 4 : *Graph of $x \mapsto x^{2n}$ and algebraic surface $x^{2n} + y^{2n} + z^{2n} = 1$*

In fact, any even number $2n$ instead of 16 (starting with $n = 2$) for the exponent is also a valid choice. The higher the exponent, the more similar the algebraic surface will be to the cube. This can be seen by looking at the graph of the function $x \mapsto x^{2n}$, which looks more angled for larger n (Figure 4). Note that the exponent 2 (i.e., $n = 1$) would give a round sphere because of the 3-dimensional Pythagorean theorem.

Next we wish to construct an algebraic surface which approximates a dodecahedron. To this end we start with the three *golden rectangles*. Their 12 vertices are given by the coordinates $(\pm 1, \pm\phi, 0)$, $(0, \pm 1, \pm\phi)$, and $(\pm\phi, 0, \pm 1)$, where $\phi := \Phi - 1 = \Phi^{-1}$, and Φ is the golden ratio [11], therefore $\phi \approx 0.62$. It is well known and can be checked by basic calculations that these 12 vertices span an icosahedron [12]. The dodecahedron and the icosahedron form a *dual pair*, faces and vertices are interchanged [3]. So we *dualize* the described construction and take the coordinates of the vertices as the coefficients of the simple equation $ax + by + cz = d$, where a, b, c can be 0, ± 1, or $\pm\phi$. Such an equation describes a plane that is orthogonal to the corresponding vertex vector (a, b, c), and an antipodal pair of vertices $\pm(a, b, c)$ yields two parallel planes. If (a, b, c) is a unit vector, d measures the distance of the plane to the origin. We finally get 6 planes parallel to the 12 faces of a dodecahedron (each face is parallel to its opposite). By taking the sum of 6 such equations, and each to the power of $2n$, where $n = 8$, we get:

$$(ax+by+z)^{16}+(-ax+by+z)^{16}+(x+ay+bz)^{16}+(x-ay+bz)^{16}+(bx+y+az)^{16}+(bx+y-az)^{16} = 1$$

For $a = 0$ and $b = 0$ it is again the equation of the cube. And for different values of the coefficients a and b we obtain smooth surfaces of the shape of an octahedron (Figure 5.a), a rhombic dodecahedron (Figure 5.b), and a dodecahedron (Figure 6).



(a) $a = 1, b = 1$ (b) $a = 1, b = 0$

Figure 5 : *Octahedron and rhombic dodecahedron*

In fact, we can interpolate between different shapes in real time by using variable coefficients within the SURFER software. The authors have also created a short movie on these form transformations [13], which has been submitted to the Bridges conference 2015.

Acknowledgments. The authors would like to thank Gert-Martin Greuel, founder of the IMAGINARY project, for his vision, dedication, and contagious enthusiasm in raising public awareness for the beauty of mathematics.

Figure 6 : $a = 0.62$ *and* $b = 0$ *yields the dodecahedron*

List of Photographs

1.a Pyrite from Ampliación a Victoria Mine, Navajún, La Rioja, Spain.
Source: http://commons.wikimedia.org/wiki/File:Pyrite_from_Ampliación_a_Victoria_Mine,_Navajún,_La_Rioja,_Spain_2.jpg
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1.b Pyrite from the ancient iron mines at Rio Marina, Elba Island, Italy.
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2 Platonic solids as objects of art by Ekkehard Neumann at Steinfurter Bagno, Burgsteinfurt, Germany.
Source: http://commons.wikimedia.org/wiki/File:Platonische_Koerper_im_Bagno.jpg
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[13] Klaus, S. and Violet, B., "Katzengold", *short movie*, to appear on the IMAGINARY platform.