Knotology Baskets and Topological Maps

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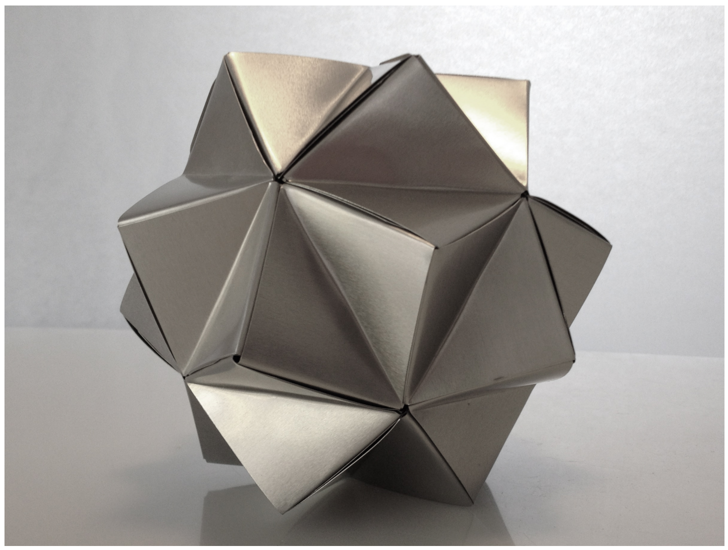

Figure 1 : *The surface of this aluminum basket exhibits knotology's corrugated kagome weaving. The basket's shape models a deltahedron (in particular, an icosahedron) that has been omni-capped with right triangles.*

Abstract

Knotology is a new type of weaving invented by Heinz Strobl that combines tabby weave and a corrugated form of kagome. The weaving elements are rectangular strips of standard dimensions, creased in a repeating pattern of right triangles. The surfaces of knotology baskets are composed entirely of squares that are diagonally creased or folded. From such simple means great variety is achieved. This paper examines the scope of this new type of weaving. I derive a correspondence between knotology baskets and topological maps, establishing the scope of knotology weaving at a topological level, but leaving unresolved the crucial issue of geometric realizability, i.e., whether such a basket has at least one conformation in three-dimensional space without forbidden folds. Experiments with paper models corresponding to some of the smallest topological maps are reported.

Introduction

In the 1990's Heinz Strobl [1] invented a new type of weaving he called *knotology.* Knotology reconciles the two fundamental ways to weave (see Figure 2): tabby weave and kagome. The weaving elements that achieve this alchemy are straight, standard-width, rectangular strips that are creased in an unvarying sequence of right triangles (Figure 3). Remarkably, nearly 100% fabric coverage is achieved (the characteristic hexagonal 'eyes' of the kagome weave are closed up.) Though knotology is limited to thin materials, like paper or sheet metal, that can sustain a sharp bend, it achieves surprising versatility from its standard parts.

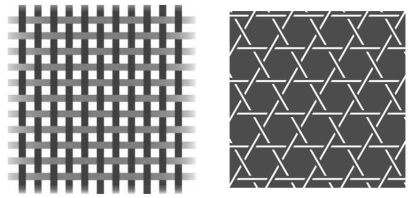

Figure 2 : *The two basic types of weaving are tabby weave (left) and kagome (right.)*

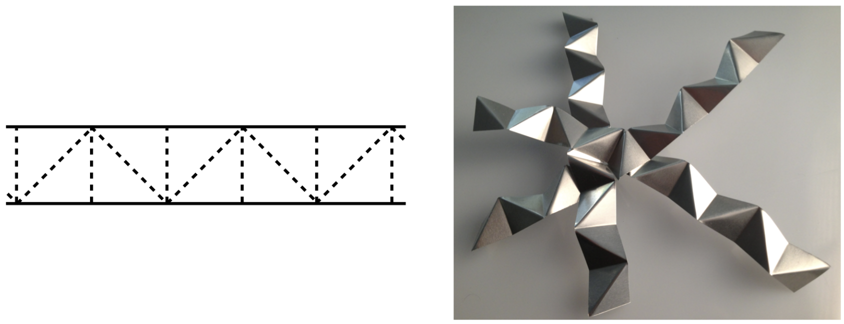

Figure 3 : *A knotology weaving element (left) is always creased in a pattern that might be described as a triangular wave superimposed on a strip of squares. Whether a given crease becomes a mountain fold, a valley fold, or is not folded at all, depends on the shape being woven. At right: pre-folded aluminum weaving elements like those used in the basket in Figure 1. The necessary splices (usually simple overlaps) are strategically placed where they will not show on the surface.*

Motivation. The great variety of knotology weaving may be glimpsed from the fact that it can make all the voxel-based shapes in computer science, and also stellated (i.e., omnicapped) versions of all the deltahedra—shapes that are duals to the trivalent nets studied in theoretical carbon chemistry [2]. (For example, the basket in Figure 1 can be understood as a model of the fullerene C_{20}.) The goal of this paper is to delimit the scope of knotology baskets. In order to emphasize the naturalness of Strobl's invention, our approach will be to start from a purely mathematical ambition to assemble surfaces from quadrilaterals and squares, and later introduce the correspondence to weaving and knotology.

Building Topological Surfaces from Quadrilateral Tiles

Topological maps [3] are of interest to sculptors and weavers because they correspond to all the ways *graphs* (for our purposes, connected line-and-dot drawings) can be *properly* drawn on closed surfaces (to be properly drawn, a graph drawing must cut the surface into faces that are topological disks,) considered up to topological equivalence. More than *drawing* on surfaces, sculptors and weavers are interested in *building* surfaces from identical parts. Because a graph properly drawn on a closed surface implicitly partitions the surface into topological quadrilaterals (see Figure 4,) we may expect that topological maps have a correspondence to the ways closed surfaces can be assembled from quadrilateral pieces or *tiles.*

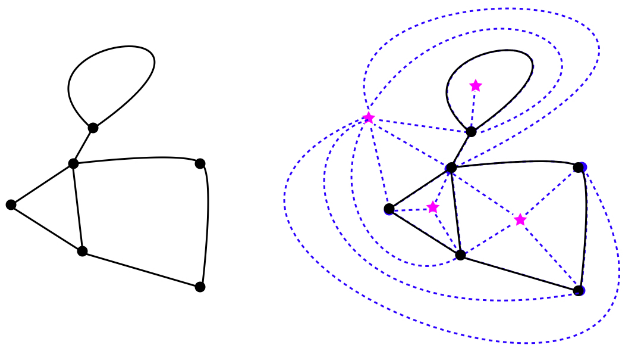

Figure 4 : *A graph properly drawn on a surface (left) implicitly partitions the surface into topological 4-gons—one for each edge (dashed blue lines at right.) (To be properly drawn on a surface, a graph must cut the surface into faces that are topological disks—thus this graph should be understood as drawn on the surface of a sphere.)*

A Bijection from the Topological Maps

When sculptors and weavers attempt to build surfaces out of identical quadrilateral tiles, they are in effect seeking solutions to a contrived mathematical game that we will call *Quadtiles*...

The *Quadtiles* game.

GAME PIECES: *Quadtiles* (topological 4-gons.)
NUMBER OF PIECES IN SET: Unlimited.
GOAL: Join quadtiles edge-to-edge to assemble a closed surface.

Every solution to *Quadtiles* corresponds to a topological map: each pair of mated tile edges corresponds to a map edge, and each cluster of tile corners corresponds to a map vertex. Therefore, every solution to *Quadtiles*—solutions for both orientable and non-orientable surfaces—can be found in The *Atlas of Topological Maps*[1]. The solutions appear in the *Atlas* as the maps that have all quadrilateral faces. For example, Figure 5 illustrates a map on the torus that provides a *Quadtiles* solution.

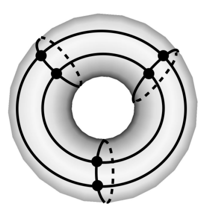

Figure 5 : *A* Quadtiles *solution on the torus.*

The bicolored radial construction. The construction in Figure 4 suggests a different way that we might use the *Atlas of Topological Maps* to find *Quadtiles* solutions. We formalize that construction as the *bicolored radial construction:*:

> In a proper graph drawing having black-colored vertices, insert a new white-colored vertex in the center of each face; construct an edge from the new white vertex to each original black vertex

[1]Jackson and Visentin have published the first pages of this infinite book [4].

> incident to that face (i.e., a new edge for each time a given black vertex is incident to that face); delete all the old edges.

Applying the bicolored radial construction converts a graph properly drawn on a surface it into a mesh of quadrilaterals whose graph has the property of being bipartite: namely, its vertices can be 2-colored such that no edge has the same color at both ends; and, in fact, the mesh of quadrilaterals comes already equipped with such a bicoloring. The edges of the original map haunt the radial map as undrawn black/black face diagonals—and similarly, the undrawn white/white face diagonals would be the edges of the dual to the original map.

The inverse bicolored-radial construction. There is an *inverse* construction that can undo the bicolored radial construction:

> In each face of a black-and-white bicolored graph properly drawn on a surface and having all quadrilateral faces, construct a diagonal edge (or a self-loop in the case of a doubly-incident vertex) connecting the two black vertices incident to that face; delete all other vertices and edges.

The existence of the inverse bicolored radial construction shows that we are in possession of a bijection between maps and bicolored, quad-faced maps.

Quadtiles under Bicolor Rules. We now introduce a restricted form of play in *Quadtiles* called *Quadtiles under Bicolor Rules,* where the quadrilateral tiles are no longer blank, but properly vertex 2-colored, that is, the tiles have corners that have been colored in this cyclical order: black, white, black, white. Also, a new rule is added: *tiles must be assembled so that corner colors match.*

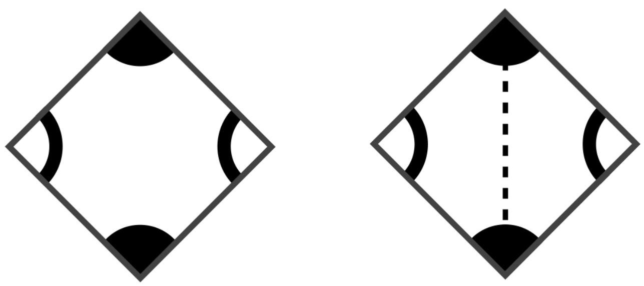

Figure 6 : *Game pieces. Left, a tile marked for Quadtiles under Bicolor Rules (the same markings are later used for the Bi-Squares game.) Right, in Black-Hinge Bi-Squares the tiles are hinged along the black/black diagonal.*

***The Atlas of Topological Maps* is the solution book for *Quadtiles under Bicolor Rules*.** It follows from the bijection derived above, that every *Quadtiles* solution under *Bicolor Rules* is coded by a topological map. Namely, for every map the bicolored radial construction draws a picture of a solution to *Quadtiles under Bicolor Rules,* complete with tile edges and matched corner colors. We know there can be no other solution, because, supposing there were, the inverse construction would trace it back to a topological map, contradicting the supposition. The *Atlas of Topological Maps* is therefore the solution book to *Quadtiles under Bicolor Rules.*

What blank-tile solutions are ruled out by *Bicolor Rules?* Some solutions to *Quadtiles* do not satisfy *Bicolor Rules.* For example, the toric map in Figure 5 contains a 3-cycle, so it cannot be bicolored. On the sphere, however, there are no such cases: every quad-faced spherical map is bicolorable [5].

Building Geometric Surfaces from Square Tiles

Having in hand a large collection of *Quadtiles* solutions, let's geometrize the play with *Bi-Squares*...

The *Bi-Squares* game.

> GAME PIECES: *Bi-squares* (rigid squares marked for play under *Bicolor Rules*)
> NUMBER OF PIECES IN SET: Unlimited.
> GOAL: Join bi-squares edge-to-edge (making hinged joints) while matching corner colors to assemble a closed surface.

We already have in hand all the ways the edges of color-matched bicolored squares can be mathematically identified to compose a closed surface; but a doubt remains: will our solutions have geometric realizations in three-dimensional space when the topological quads are replaced by rigid geometric squares?

Voxel-based surfaces are geometrically realizable solutions to *Bi-Squares.* Reassuringly, there is a large class of bipartite, square-faced surfaces that do have geometric realizations in three-dimensional space. The vertices of the 3D packing of (hollow) unit cubes can be properly bicolored by coloring each vertex according the parity of the sum of its integer coordinates, $(x + y + z)$. Therefore, any closed surface composed of unit squares selected from this packing (e.g., the bounding surface of a solid composed of unit cubes, or *voxels*) is a solution to *Bi-Squares*—one that comes with a ready-made geometric realization.

Paper Model Experiments

OK, but what happens in the case of ordinary maps? The chart in Figure 7 summarizes experiments with paper bi-squares assembled as directed by some of the smallest topological maps; namely, all of the maps on the sphere having three edges or less, plus the simplest toric map. Since maps in a dual pair direct the same physical assembly of tiles (the only difference being a rotation of the colors black and white) dual pairs are displayed on the same row in the table. The first digit in the map number (as assigned in [4]) indicates the genus of the map. The model tests show that only m0.5 supplies a geometrically realizable solution to *Bi-Squares*—all the other models need to fold at least one square in order to assemble as their map directs.

Adding Diagonal Hinges

Diagonal hinges are a practical necessity. Though we have all the *topological* solutions to *Bi-Squares,* experimentation shows that few of them actually have a realization when geometric squares are used instead of topological quads. We are forced to make the game easier by allowing bi-squares to bend. For the bending to be geometrically determined, it needs to be concentrated at a hinge. Limited to a square's four vertices, there is only one place to insert a hinge: along one of the diagonals.

Which diagonals should we hinge? Each bi-square has two diagonals, as designated by the color of their endpoints, a white diagonal and a black diagonal. Which should we hinge? The extravagant option of hinging *both* diagonals would, in effect, *refine* our graph, adding four edges and a vertex instead of adding a single edge. The custom-folded option of hinging whichever diagonal is most advantageous at a particular location would add an information overhead to the construction process and endanger our scheme of building from identical parts. The only choice remaining is between all-black or all-white hinging.

Black hinges are the sensible choice. Because the black diagonals correspond to the edges of the original map, it is sensible to make black hinges canonical—and thus make the edges of the original map visible on the assembled surface as the hypotenuse folds. (In any case, the same assembly of squares and hinges offered by white hinges can be obtained by using black hinges with the dual map.)

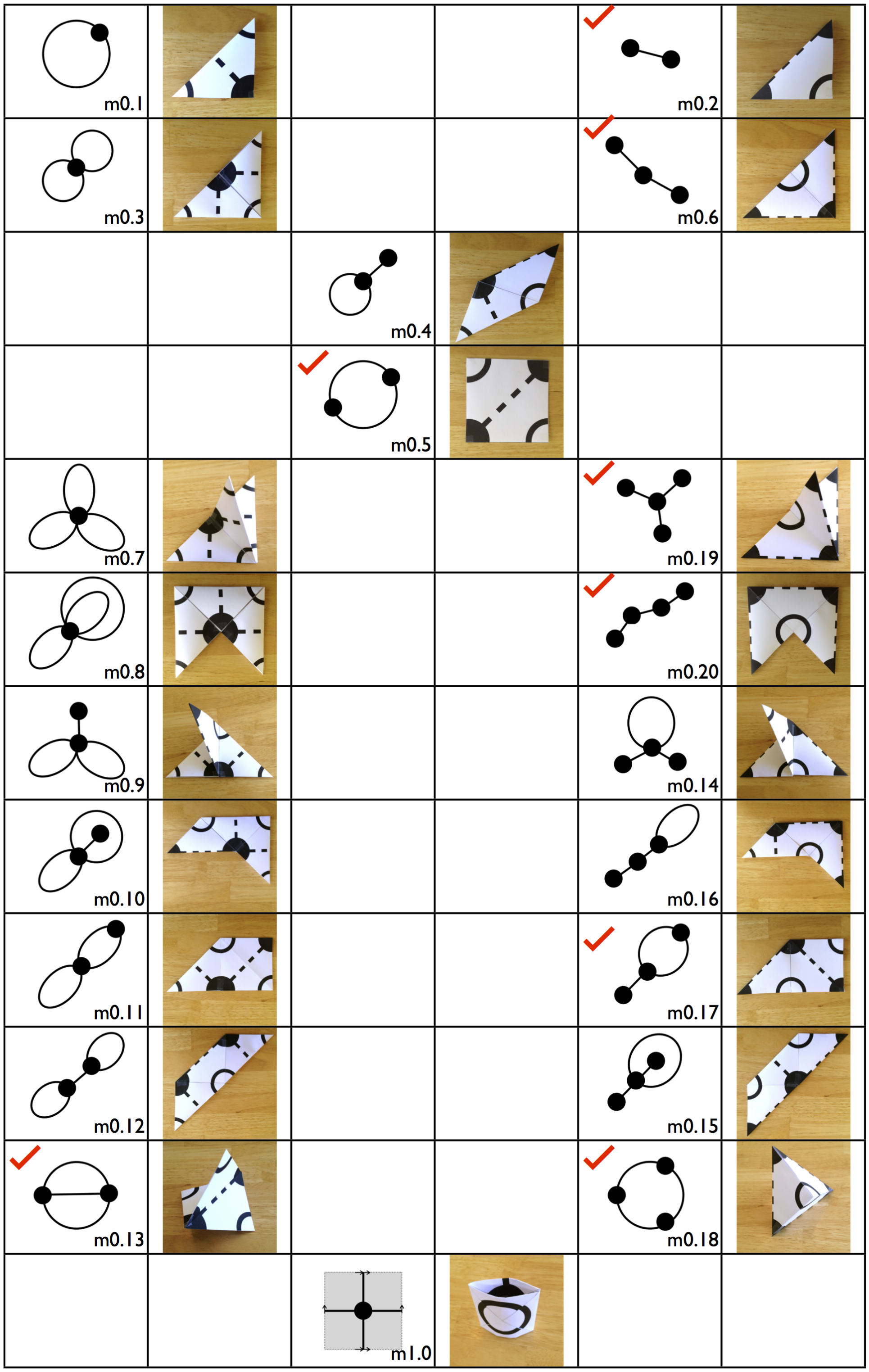

Figure 7: *The Bi-Squares solutions corresponding to the smallest maps on the sphere and the smallest map (m1.0) on the torus. Black hinges allow the check-marked maps to have a geometric realization—all the other paper models have illegal folds. One map, m0.5, needs no hinges at all. Map atlas numbers from [4].*

Black hinges permit eight out of the twenty-one models tested to pass. The models that still fail to have a geometric realization after black hinges are added all contain self-loops. Since an edge must fold along some axis not collinear with itself in order to close a self-loop, a map with a self-loop is unrealizable if only black-hinges are permitted. Other causes of geometric unrealizability may arise as larger maps are tried.

Knotology Weaving

Over one, under one, over one, under one...is the sturdiest way to weave a basket. Computer scientists call this pattern *plain weaving* [6], a term we will adopt here, though it generally carries a narrower meaning in the textile arts.

Tait's checkerboard graph. An early idea in knot theory has only more recently been brought into practical application by artists [7, 8]. In 1876, P. G. Tait [9] observed that plain weaving can be represented by a 2-coloring of the faces of a 4-regular graph drawing (4-regular means every vertex is incident to four edges.) Such a coloring inevitably suggests a chessboard or checkerboard. This insight originated in the observation that every opening in a plain weave can be assigned a left- or right-handedness—and never does the same handedness border both sides of a weaving element.

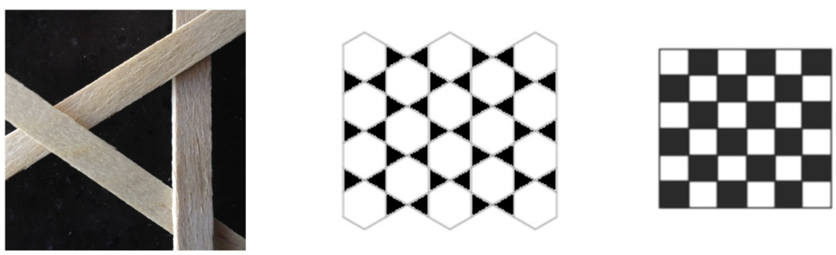

Figure 8 : *Left: a weave opening in a plain-woven fabric gives the appearance of an impossible staircase. A bug going around this opening in the counterclockwise direction would seem to be climbing stairs. Associating curled fingers with the direction of orbit, and an extended thumb with the direction of apparent vertical progress, we can classify this weave opening as right-ISH. Under the ISH color convention this opening would be colored black. Center and right: checkerboard graphs for the two basic plain weaves, kagome and tabby weave.*

The ISH method of assigning handedness. Figure 8 shows an easy method to assign handedness to a fabric opening. Handedness determined by this method will be called *impossible staircase handedness,* (ISH.) Importantly, assignment of handedness does not depend on which side of the fabric we are looking at. We also adopt an *ISH color convention* associating the color black with right-ISH, and white with left-ISH. The right side of Figure 8 shows examples of checkerboard graphs for the kagome and tabby weaves. Under the ISH color convention there is only one way to weave the illustrated examples.

The dual representation of a checkerboard graph. A checkerboard graph is a 4-regular graph drawing equipped with a proper bicoloring of its faces in black and white. The same information can be conveyed by its bicolored dual graph, namely, a bipartite, quad-faced graph equipped a bicoloring of its vertices in black and white, where each *vertex* in the dual comports in color with the corresponding *face* in the checkerboard graph. Through this dual representation of the checkerboard graph, *every surface assembled from bi-squares explicitly describes a weaving pattern.*

Knotology weaving. Knotology weaving uses weaving elements (weavers) that are straight, constant-width, and creased in the pattern illustrated on left side of Figure 9. As shown, a straight run of black-hinge bi-squares possesses an arrangement of hinges that perfectly mimics the creases of a knotology weaver.

Weaving from a Black-Hinge Bi-Squares solution. Given a closed surface assembled from bi-squares, it is easiest to imagine weaving the corresponding knotology basket around it. In this process, each bi-square will be overlaid by two knotology weavers crossing at right angles to each other and parallel to the sides of the bi-square. The diagonal creases of both weavers will overlie the black hinge of the bi-square. At the places where weavers engage at their edges, a small window is left in the woven fabric. These *engagement windows* overlie the vertices of the bi-square. The color of the vertex underneath an engagement window determines the ISH of the weave opening, and thus determines which weavers go over and under at the crossing.

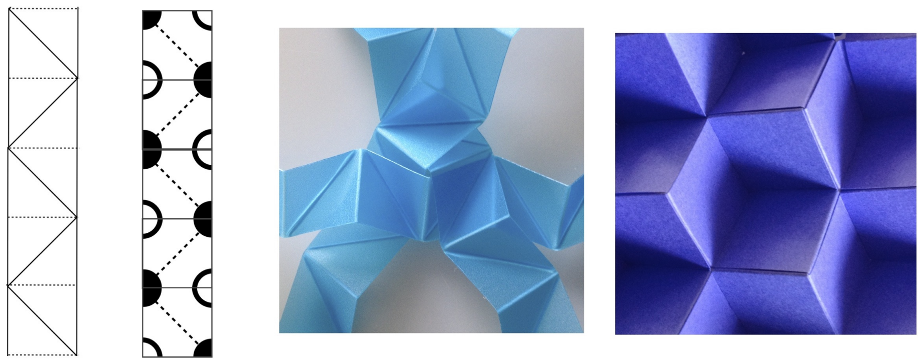

Figure 9: *Left: a knotology weaver compared with a straight-line play of black-hinged bi-squares. Center: three knotology weavers forming a right-ISH (black) vertex. Right: a view of a woven voxelized surface that approximates a plane perpendicular to the* $(1, 1, 1)$ *vector.*

Conclusion

Every topological map describes a knotology basket, but in many cases the basket will not have a geometric realization unless non-standard folds are added. A characterization of which topological maps correspond to geometrically realizable knotology baskets would be desirable. The author is grateful for the reviewer's suggestions.

References

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