Nonplanar Expansions of Polyhedral Edges in Platonic and Archimedean Solids

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Abstract

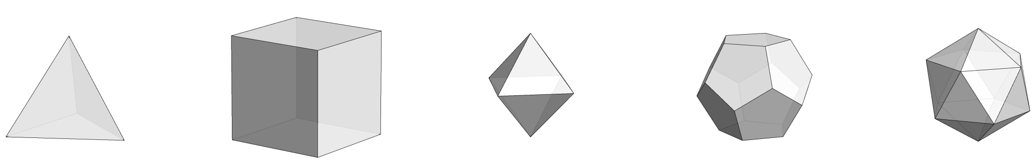
The process of replacing each edge of a regular polyhedron with a square results in the creation of a new object, similar to the process of Stott expansion. However, following the edge to square transformation, the resulting object's surface no longer has genus zero. In some cases, the object also contains bumps or craters to accommodate the additional length of material. This process can be generalized to any polyhedral form having equal length edges, such as Platonic solids, Archimedean solids, prisms, and anti-prisms. Examples are shown for these particular classes of polyhedra using a variety of materials and symmetries.

Background

Platonic solids are some of the most basic bounded three-dimensional objects, having equal regular (equilateral and equiangular) polygon faces with vertices all lying on a sphere. The earliest written account of these objects occurs in Plato's *Timaeus*. They are discussed in detail in Euclid's *Elements*. There are five such solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron as illustrated in Figure 1.

The Archimedean solids are polyhedra in which the faces are regular polygons and the vertices are surrounded by the same number and type of regular polygons. There are thirteen such solids as shown in Figure 2. Like the Platonic solids, the objects exhibit a high degree of symmetry. Pappus of Alexandria, in the fifth book of his *Mathematical Collection*, credits this set of objects to Archimedes. Sadly, the Archimedes writing assumed to be used by Pappus is not extant. The account of these solids by Pappus only lists the number of faces of each polygon type. Kepler was able to recreate them by the early 17th century and gave them their common names [1].

The Archimedean solids can all be derived from the Platonic solids with a few basic transformation operations. For example, truncating the corners of each Platonic solid until the resulting edges are equal in length results in the truncated tetrahedron, truncated cube, truncated octahedron, truncated dodecahedron,



Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
4F, 4V, 6E	6F, 8V, 12E	8F, 6V, 12E	12F, 20V, 30E	20F, 12V, 30E
3.3.3	4.4.4	3.3.3.3	5.5.5	3.3.3.3.3

Figure 1 : *Platonic solids. The five Platonic solids are polyhedra having equal regular polygon faces with vertices all lying on a sphere. Below each image is a list of the number of faces (F), vertices (V), and edges (E) along with the vertex type.*

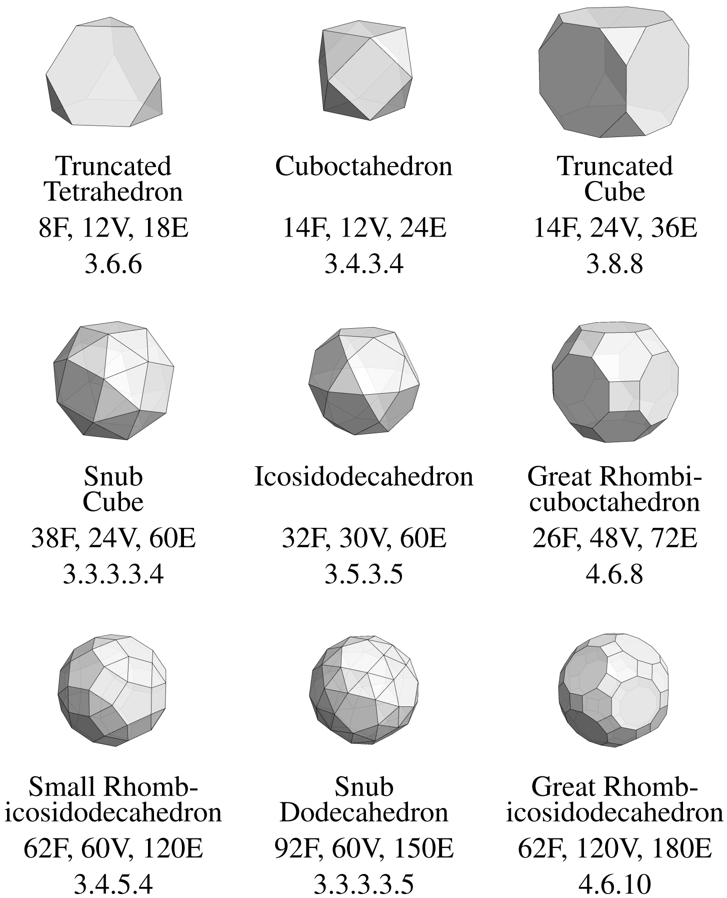


Figure 2 : *Archimedean solids. The Archimedean solids are polyhedra in which the faces are regular polygons and the vertices are surrounded by the same number and type of regular polygons. The number of faces (F), vertices (V), and edges (E) along with the vertex type is also given.*

and truncated icosahedron. Likewise, the cuboctahedron can be obtained by truncating the cube or octahedron until the truncation planes intersect at the middle of the original edges. Placing equilateral triangles around the faces of a cube and dodecahedron results in snub version of each; these are the only Archimedean solids that are chiral.

Another form of transformation is called Stott expansion [4]. In this expansion process, faces are pushed outward from the center of a polyhedron, leaving a gap between edges. The resulting gap at a former edge can be filled with a rectangular polygon, leaving a polygonal void at the location of the original vertices. This can also be filled with a polygon, closing the object. The Archimedean solids that have mirror symmetry can all be obtained from the Platonic solids through a process of expansion [5].

In researching potential classroom projects, I recently came across an interesting object on the *Instructables* website called a "Windball" [2]. Descriptive instructions for making it were posted by Makedo around 2011 and are attributed to Tanaka Satoshi. The design calls for thirty cardboard squares connected at the corners by special fasteners and makes a ball several feet in diameter that has a high degree of symmetry. I decided to make one using small pieces of paper and split-pin fasteners. Squares were arranged such that five squares were connected resulting in pentagonal openings. Squares were added to the unconnected corners resulting in triangular openings. This process was repeated in such a way as to result in roughly polyhedral object. The flex of the paper allowed the final closing of the object into a roughly spherical shape.

Upon closer inspection the Windball (see icosahedron model in Figure 3) was seen to be roughly the small rhombicosidodecahedron (see Figure 2) with paper for each of the square faces and voids for the triangular and pentagonal faces. The Windball can also be seen as a dodecahedron with the edges replaced by squares, the faces transformed into pentagonal openings, and the vertices transformed into triangular openings. It can also be seen to be equivalent to an icosahedron in which the edges have been replaced by squares. Thus, this object shows both the icosahedron and the dodecahedron simultaneously because they are duals of one another.

Further search for similar objects revealed a set of four similar objects called "square unit spheres" constructed by Yoshinobu Miyamoto [3]. As with the Windball, the square unit spheres are created by replacing edges of a polyhedron with squares. In this set, the objects were constructed from paper squares connected at their corners with slots. In these objects, squares replace edges of a tetrahedron, a cube/octahedron, and a dodecahedron/icosahedron. In the fourth object, the underlying polyhedron is simultaneously the icosidodecahedron and its dual, the rhombic triacontahedron. However, there is no explanation on how these objects are created and no generalization to other possible configurations.

In these examples (Windball and square unit spheres), the solid square faces are very close to being planar. If the squares are allowed to bend, then squares in other configurations can be fastened together while maintaining squares that are roughly plane-tangent at their corners.

Another similar set of objects, Juno's Spinners, use polygonal shapes connected by struts that allow the entire shape to rotate and expand [6]. The polygons remain essentially planar while the struts flex.

Replacing edges by squares

The process of replacing each edge of a regular polyhedron with a square results in a new object, similar to the process of Stott expansion. Each square must bend, becoming nonplanar, in this expansion process. The original faces and vertices are replaced with open voids. Thus the resulting object's surface no longer has genus zero, it has genus $F + V - 1$, where F and V are the number of faces and vertices of the original polyhedron respectively. This process can be generalized to any polyhedral form having equal length edges, such as Platonic solids, Archimedean solids, prisms, and anti-prisms. With some polyhedra, such as the icosahedron (see Figure 3), this results in an object where the squares have little planar deformation. In other cases, such a the tetrahedron (see Figure 3), the squares become cylindrical patches. Polyhedra with less uniform vertices, such as the truncated cube (see Figure 4), have some squares with positive curvature and others with negative curvature, thus the radial distance from the center to the surface can have a wide inter- and intra-square variance.

Expanded models based on the five Platonic solids were constructed using laser-cut paper-backed bamboo veneer and illustrated in Figure 3. Each square was precisely cut to have an edge length of 2.125 in. Because of the orientation dependent stiffness of the material, orientation of the grain in the direction of the original edge results in an object more closely resembling that polyhedron rather than its dual, despite having exactly the same placement and connections of the squares. Starting with the tetrahedron and replacing each edge with a square, the resulting object has six squares and six triangular openings. Because the tetrahedron is self dual, triangular openings can be considered either the faces or the vertices of the underlying tetrahedron. Replacing each edge with a square in the cube and octahedron results in the same object because they are duals. The number of edges in each base polyhedron is twelve, requiring twelve squares. A square opening corresponds to the faces of the cube and vertices of the octahedron. Triangular openings are also possible, corresponding to the faces of the octahedron and vertices of the cube.

This process was then repeated for each of the thirteen Archimedean solids. The resulting objects are

Figure 3: *Expanded Platonic solids. Example models of the Platonic solids after a nonplanar expansion of the edges by replacement with a square. The models were constructed using laser-cut paper-backed bamboo veneer squares connected using split-pin fasteners. Each square was 2.125 in $\times$ 2.125 in. Solid models of the base polyhedra are shown below the expanded models. The Windball described in the text is similar to the expanded dodecahedron and icosahedron.*

shown in Figure 4. As with the expanded Platonic solids, these objects map both faces and vertices to openings. This results in the simultaneous appearance of the Archimedean solid and its dual Catalan solid.

The squares used in these models were hand cut from index cards on a table-top paper cutter. The squares were nominally 2 inches on a side, but might vary in some dimensions by around an eighth of an inch. Additionally the holes in the paper were hand punched using a paper punch with a 1/16 inch hole size. No specific provision was made to have the holes located a precise distance from the corners. Thus the variability in the size of the square and the variability in the position of the holes relative to the squares resulted in a roughly square shape among the four holes on the square. This was done randomly and with every square comprising the object. Thus the variability averaged out to give the overall appearance of a symmetric object, especially in large objects with many squares.

Substituting a two-dimensional square for a one-dimensional linear edge results in the object no longer having vertex points all lying on the surface of an enclosing sphere. The result is that some edges are closer to the center of the sphere while others are further away. This manifests itself as outward bumps or craters on a roughly spherical object. With a highly symmetric polyhedra like Platonic and Archimedean solids, these craters or bumps are uniformly or symmetrically distributed on the surface of the object. Such bumps are noticeable where there is an isolated triangle, as in the expanded truncated Platonic solids. In the expanded snub polyhedra, the resulting bumps and craters result in a puckered surface that does not closely resemble a sphere. The expanded great rhombicuboctahedron found a stable configuration with some organically symmetric regions pushed inward or outward, resulting in a lobed tetrahedral form.

The bending axis of the squares is oriented with the square's edges with the exception of the snub polyhedra, where the bending is aligned along the diagonally opposite vertices of the square.

Replacing edges by other shapes

One generalization of this expansion process replaces each edge with a rectangle. Objects were created using standard business cards for the expanded cuboctahedron using twenty-four cards as shown in Figure 5, and for the expanded icosidodecahedron using sixty cards as shown in Figure 6. While a blank standard business card has D_2 symmetry, the printing results in an asymmetric edge. A highly symmetric object with respect to the printing is still possible with the cube, octahedron, and some of the Archimedean solids. Determining which symmetries are admissible is a subject of future research.

Figure 5 provides a visual comparison for using different edge types and orientations in creating an expanded cuboctahedron. Figure 6 provides two examples of an expanded icosidodecahedron and an expanded small rhombicosidodecahedron.

Replacing edges by other closed shapes is possible as illustrated in Figures 5 and 6. A laser-cut paper 60° section of an annulus (having mirror symmetry) was used. In Figure 5, the annulus section was oriented in three different ways. The first resulted in a chiral structure, with 3-fold and 4-fold pinwheels. The other two structures had mirror planes and contained six and eight circular regions, echoing the faces of the cube and octahedron respectively. In Figure 6, sixty annulus sections were arranged in a chiral pattern following the edges of the icosidodecahedron, resulting in 5-fold and 3-fold pinwheels.

The final example shown in Figure 6 is an artwork entitled "Inconceivable Symmetries" that is an expanded small rhombicosidodecahedron. Each edge of the small rhombicosidodecahedron has been replaced by two packed colored condoms placed back-to-back and connected with split-pin fasteners. Each package is roughly square and has no preferential bending direction.

Construction Issues

As seen in the examples, a variety of materials can be used to construct these expanded polyhedral forms. Paper is a nice material to use because it is widely available in a wide range of colors and is easily manipulated. It also has an isotropic bending preference so that the bend in the material is independent of the direction of the square placement. Generally the material should be flexible enough to conform to the dihedral angle between the corner-connected expanded edges, keeping the connected corners roughly in the same tangent plane. This is not a problem in a conventional polyhedron because the surface is not differentiable at its vertices. At the same time, the material must be stiff and light enough to support the object without bending too far. Even with 110 pound paper, some of the larger models were not fully self-supporting.

A variety of types of fasteners can be used to connect the expanded edges. Prior works were constructed using tabs or slots cut into the corners of the paper squares. As with the edge material, the fasteners should be lightweight and be strong enough to hold the object together under its own weight. In my work I have used small split pin fasteners. These fasteners are placed in small holes that are punched or laser cut into the material. Each base edge is replaced by an expanded edge that has four connectors, and each corner is connected to a corner of a different expanded edge. Thus the number of fasteners needed is equal to the twice number of original edges.

The construction process with split-pin fasteners and small paper can be challenging and time consuming. After much skilled construction practice making many of the objects, my average assembly time is about one minute per connector. This process can be simplified by following the symmetry of the underlying object. For example, when creating a pentagonal opening, the five squares needed can be placed such that the edge to the left is always on top of the edge to its right or vice versa. In this way modules can be created with pentagonal openings and then stuck together when the underlying objects permits this. In other cases one must start with one type of opening and then build the object from there. When constructing the expanded great rhombicosidodecahedron (having 180 edges), the twelve dodecagonal openings were all constructed, then the interwebbing of squares was constructed. Then the dodecagonal units were fitted into the openings on the larger object. The challenge of finding symmetric modules in such an object adds to the fun of constructing them.

Discussion and Conclusions

The process of edge expansion was applied to Platonic and Archimedean solids to construct objects with a lattice surface. A variety of materials and symmetries were used for the expanded edges, resulting in elegant symmetric forms.

There is a interesting trade-off between fastener type, material rigidity, and object size. The larger the squares used, the stiffer the material must be for the object to hold its shape. However the material must not be so rigid that it does not allow the flexibility needed to form to the needed shape.

Reducing the variability in the size of the expanded edges and the hole placement results in forms with improved symmetry. For example, when using the paper-backed bamboo veneer, the squares and holes were laser-cut, resulting in very precise control over the dimensions and placement of the holes. The resulting shapes end up being very precise and noticeably more symmetric than the hand cut hand punched squares initially used.

Future plans include constructing the entire set of expanded Archimedean solids using paper-backed bamboo veneer and other interesting materials. While not shown here, this method can be used for other polyhedra with uniform edge length, such as prisms, anti-prisms, and Johnson solids. I plan on also exploring using expanded edges with other simple geometric forms having different symmetry types.

Acknowledgments

I am grateful to Albion College professors A. French for donating the bamboo veneer and A. Miller for use of the laser cutter. I am also thankful to the constructive feedback provided by the anonymous reviewers.

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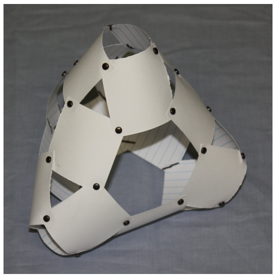
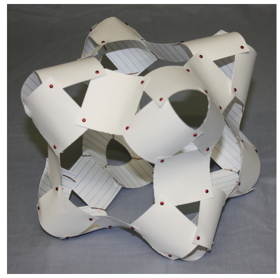
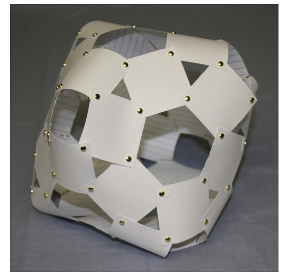

Truncated Tetrahedron


Cuboctahedron


Truncated Cube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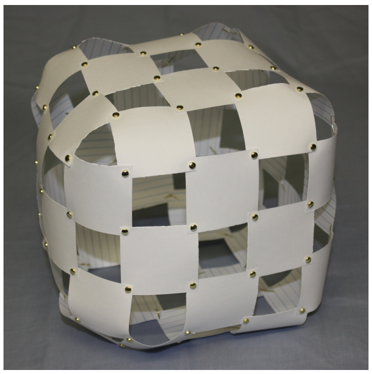
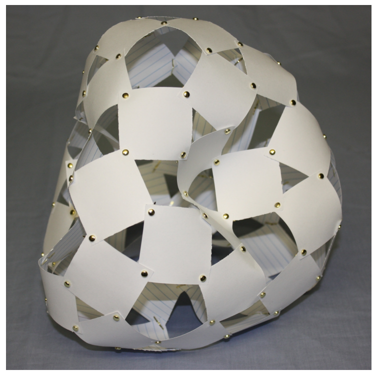

Small Rhombicuboctahedron


Snub Cube


Icosidodecahedron


Great Rhombicuboctahedron


Truncated Dodecahedron


Truncated Icosahedron


Small Rhombicosidodecahedron


Snub Dodecahedron


Great Rhombicosidodecahedron

Figure 4: *Expanded Archimedean solids. Example models of the Archimedean solids after a nonplanar exansion of the edges by replacement with a square.*

Figure 5 : *Expanded Cuboctahedron patterns. Example models of the Cuboctahedron after a nonplanar expansion of the edges by replacement with several objects illustrating a range of possible edge types. An expansion by squares (having D_4 symmetry) is shown. Two examples using standard business cards (having D_2 symmetry ignoring writing), oriented $90°$ from one another are shown. Three models using a $60°$ section of an annulus (having D_1 symmetry) are shown: one uses a chiral pattern and the other two are oriented $90°$ with respect to one another.*

Business Cards | Annulus Segments | "*Inconceviable Symmetries*"

Figure 6 : *Example expanded Archimedean solids. Example models of the icosidodecahedron and the small rhombicosidodecahedron after a nonplanar exansion of the edges. The left model is an expansion of the icosidodecahedron using business cards with hand-punched holes. The middle model is an expansion of the icosidodecahedron using a $60°$ section of an annulus; laser-cut 110 pound paper was used. The final model on the right, an artwork entitled "*Inconceviable Symmetries,*" is an expansion of the small rhombicosidodecahedron using packaged condoms.*