From Sangaku Problems to Mathematical Beading: A Hands-on Workshop for Designing Molecular Sculptures with Beads

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Abstract

In this workshop we will start from an ancient Sangaku problem about a large central ball covered by thirty small balls. We will illustrate the problem with physical model constructed by mathematical beading. In this workshop, participants will learn the basic techniques of mathematical beading and make bead models of C_{20} and C_{60}, which correspond to Goldberg polyhedra with chiral vector (1,0) and (1,1).

In the isolated Edo period (1603-1867) of Japan, new kind of geometric problems quite different from the Euclidian geometry become common practices from all walks of life. These problems are gradually forgotten in Japan under the influence of Western culture and science. Fortunately, in the late 60s, a high-school mathematics teacher, Fukagawa Hidetoshi, discovered accidentally this kind of interesting traditional Japanese mathematics. Since then, whenever available, he went to different temples and shrines and tried to collect and organize these problems. Finally, together with Tony Rothman, a beautiful book on Japanese Sangaku problems[1] is published in 2008. Most of these Sangaku problems are now collected in the website on Japanese traditional mathematics, http://www.wasan.jp/.

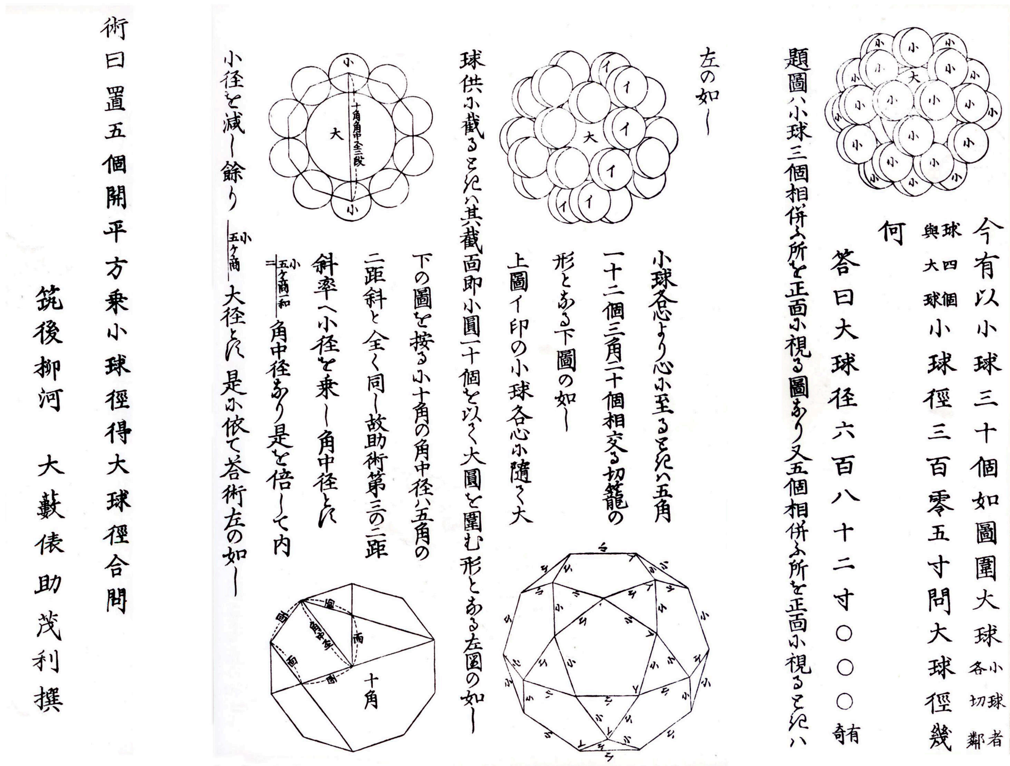
In this workshop a famous Sangaku geometric problem as shown in Figure 1 will be presented and participants will have a hands-on experience of making a model of this problem with the techniques of mathematical beading. This problem was originally described in the appendix of a book entitled "The Handbook of Mathematics (算法助術)"[2] published 1841. The original problem was written mostly in Kanji, which can be easily understood for people who understand in Chinese. Literally, this problem can be translated as:

Thirty small balls surround a central big ball such that every small ball is tangent to four other neighboring small balls and the central big ball. If the diameter of small ball is 305 in, what is the diameter of the central big ball?

Answer: 682 in.

After giving this answer, there begins a discussion on the method used to reach this conclusion, which will be explained later. Finally, a general answer $R/r = \sqrt{5}$ *is given at the end.*

The structure of typical Sangaku problems can be separated into three parts. In the first part, the problem is stated, and then the specific answer is given in the second part. Finally, the detailed procedure and the general solution are outlined. This kind of style of presenting the mathematical problems and their solutions is common in many ancient Chinese mathematical works such as "The Nine Chapters on the Mathematical Art (九章算數)" published more than two thousand years ago.


今有以小球三十個如圖圍大球小球者各切鄰球四個與大球小球徑三百零五寸問大球徑幾何

答曰大球徑六百八十二寸〇〇〇有奇

題圖ハ小球三個相併ふ所を正面ふ視る圖あり又五個相併ふ所を正面ふ視るときハ左の如し

小球各心より心ふ至るときハ五角一十二個三角二十個相交る切籠の形とある下圖の如し

上圖イ印の小球各心ふ随て大球共ふ截るときハ其截面即小圓一十個を以て大圓を圍む形とある左圖の如し

下の圖を按るふ十角の角中径ハ五角の二距斜と全く同し故助術第三の二距斜率ヘ小径を乗し角中径とし 二 小五ケ商一和 角中径あり是を倍して内小径を減し餘り 小五ケ商 大径とし是ふ依て荅術左の如し

術曰置五個開平方乘小球徑得大球徑合問

筑後柳河 大藪俵助茂利撰

Figure 1: A large ball covered by 30 small balls described in pages 51-53 of "The Handbook of Mathematics".

The trick for solving this problem is to recognize the centers of small balls are exactly located at thirty vertices of an icosidodecahedron (Figure 1), which is one of the thirteen Archimedean solids. An icosidodecahedron has 32 faces, 12 regular pentagons and 20 regular triangles, and 30 vertices; each of vertices shares an edge with four neighboring vertices. It is easy to see that every vertex on the icosidodecahedron is located on two regular decagons with their centers coincide with the center of the icosidodecahedron. Thus centers of the blue balls in Figure 2a form a decagon. This means that the centers of ten balls along this decagon lie in the same plane passing through the center of the icosidodecahedron as shown in the Figure 2b, which gives the cross section through the centers of these ten small balls and the central large ball.

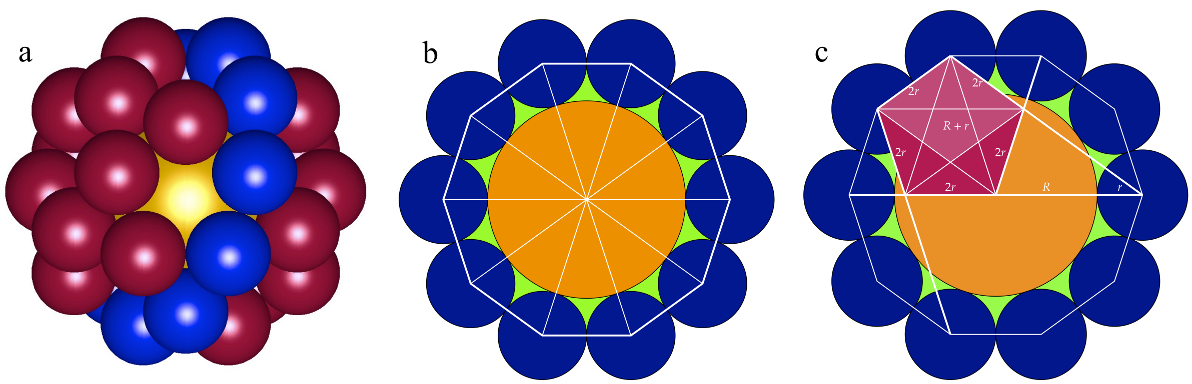


Figure 2: *Sangaku problem: 30 small balls around a central large ball.*

Two neighboring small circles (in blue) in these ten circles are tangent to each other and are also to the central large circle (in orange). It is straightforward to find the ratio between large and small circles by the trigonometry because they satisfy the identity, $\sin 18^\circ = r/(r+R)$, where r and R are radii of small and large circles. Therefore, one gets $R/r = \sqrt{5}$. However, in the Edo period, mathematicians in Japan adopted pure geometric approach without using the trigonometry. Drawing a few auxiliary lines as shown in the right figure of Figure 2c, one can identify a few similar triangles. Using these geometric relations, two identities $2r + a = R + r$ and $a/2r = 2r/(R+r)$ can be written down. Again, these equations can be solved to get the ratio $R/r = \sqrt{5}$.

In this workshop, we will use the mathematical beading to illustrate this geometric problem from the Edo period. Creating three-dimensional objects with beads has been practiced in oriental countries for centuries. The technique for making most of 3D object is the figure-eight stitch (Hachi-Ami) or right-angle weave[3,4], which will be employed to make a model consisting of 30 small beads connected by a long nylon line in exactly the same way.

To start with this hands-on project, we need to choose 30 beads with same colors. Note that we will think the 30-ball structure as a dodecahedron instead of an icosidodecahedron, which will become clear after the model is built. Thus, beads will represent the edges of a dodecahedron. Since there are twelve pentagons in a dodecahedron and each pentagon consists of five beads, we still need 12x5/2=30 beads, where a division of two is because each bead is shared by two pentagons. We will need a length of nylon fishing line approximately 80x the length of a single bead. The detailed procedure is given as follows (Figure 3a):

1. String 5 beads in the middle of the line.
2. Take the right line and insert it through the furthest bead in the opposite direction and pull the lines tight. This makes a pentagon (***1***).
3. Add 4 beads on the left line.
4. Take the opposite line and insert it through the furthest bead in the opposite direction and pull the lines tight. This makes the second pentagon (***2***). The first two loops look like a figure eight.
5. In order for the next pentagon to be fused with the existing rings, the line must exit from neighboring rings. Therefore, insert the line at the pentagon junction through the closest pentagon bead in order to make a 'V' shape - this is called *fishing* and this line is denoted as the fishing line (highlighted in grey). Note that you will always add beads to the other line (*non- fishing* line) in the following step - This allows for the symmetric growth of the fullerene according to the spiral code and simplifies the beading process.
6. Add three beads on the *non-fishing* line. Take the other line and insert it through the furthest bead in the opposite direction and pull the lines tight. This makes a pentagon (***3***).
7. Go *fishing* and add three beads on the *non-fishing* line.
8. Take the other line and insert it through the furthest bead in the opposite direction and pull the lines tight. This makes a pentagon (***4***).
9. Repeat steps **7** & **8** to make the fourth pentagon (***5***). Notice that the *fishing* and *non-fishing* lines will always alternate and both lines should decrease in length equally.
10. Go *fishing* twice and add two beads on the *non-fishing* line.
11. Take the other line and insert it through the furthest bead in the opposite direction and pull the lines tight and go *fishing*. This makes a pentagon (***6***) and completes the first set of pentagons. This structure is the hemisphere.
12. Add 3 beads on the *non-fishing* line. Take the other line and insert it through the furthest bead in the opposite direction and pull the lines tight. This makes a seventh pentagon (***7***).
13. Go *fishing* (through one, two or three beads, as necessary) and complete the remaining six pentagons around the original pentagon (**7** to ***12***) to complete the whole structure (Fig.3b).

14. *Fish* one line around the pentagon to a common vertex and tie a knot. Fish the lines again through some beads to hide the knot. Use the remaining line to hang your model or cut the lines flush to obtain a free model (Fig.3c).

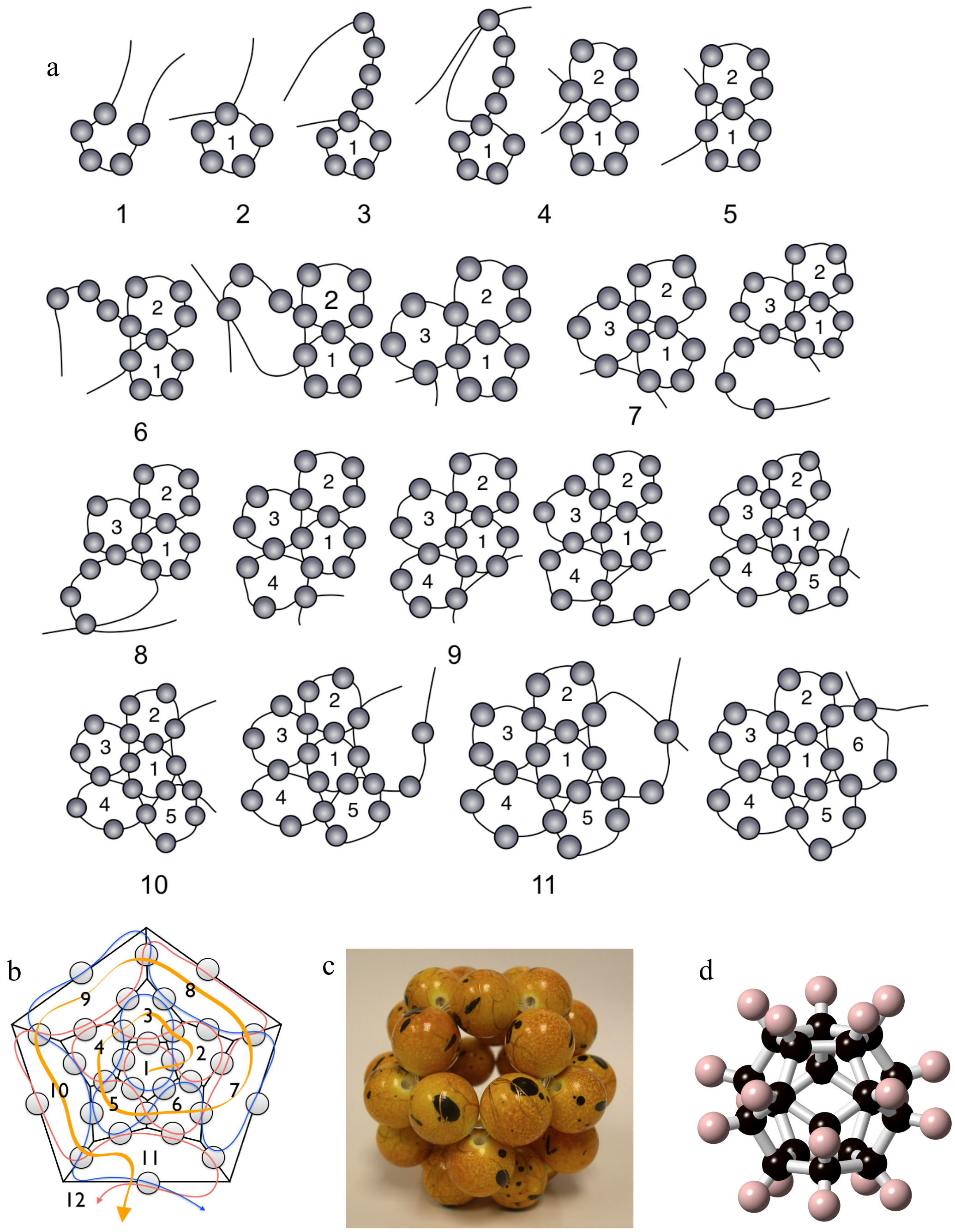


Figure 3: (a) *Detailed procedure for making the bead model for the 30-ball problem; (b) Schlegel diagram and the corresponding beading direction (thick orange line) for a dodecahedron; (c) the corresponding bead model; (d) the structure of dodecahedrane* ($C_{20}H_{20}$).

It is interesting to note that we can interpret bead models constructed with figure-eight stitch as physical models of fullerenes as long as beads represent chemical bonds instead of atoms. Therefore the bead model that consists of thirty small beads we just made is in fact corresponding to the physical model of a

C_{20} arranged as a dodecahedron. If we add additional twenty hydrogen atoms, we then get the famous dodecahrane molecule (Figure 3d). More generally, the same beading technique can also be employed to make more complicated fullerenes and graphitic structures such as C_{60}, C_{70}, carbon nanotori, carbon nanohelices, Sierpinski fullerenes, carbon Schwarzites, high-genus fullerenes, and almost any kind of trivalent carbon structures as have been shown in our publications in previous Bridges meetings[4-8].

Here we only give a brief description of using spiral codes and Schlegel diagrams to create bead models of C_{60}, C_{70} and C_{80} for the participants of this workshop. First, we show the procedure to make C_{60} (Figure 4a). If we ignore the tiny difference in bond lengths of single and double bonds, the structure of a buckyball molecule is actually the truncated icosahedron (Figure 4b), which is again one of the thirteen Archimedean solids. To make a bead model of C_{60}, it is easier to use its Schlegel diagram (Figure 4c), which gives the complete beading information. Starting from the central pentagon and then following the spiral direction specified in Figure 4c, we make a pentagon first, followed by five hexagons, and then a pentagon, a hexagon, etc. The resulting bead model is shown in Figure 4d. This sequence of pentagons and hexagons along this spiral direction can be written down explicitly as [5 66666 5656565656 6565656565 66666 5]. One can also view a C_{60} molecule as a sphere consisting of six layers of polygonal strips. For the south hemisphere, we first bead a pentagon for the South Pole, five hexagons for the next layer, and then 10 polygons consisting of 5 pentagons and 5 hexagons. For the remaining part, we reverse the beading sequence to get the north hemisphere.

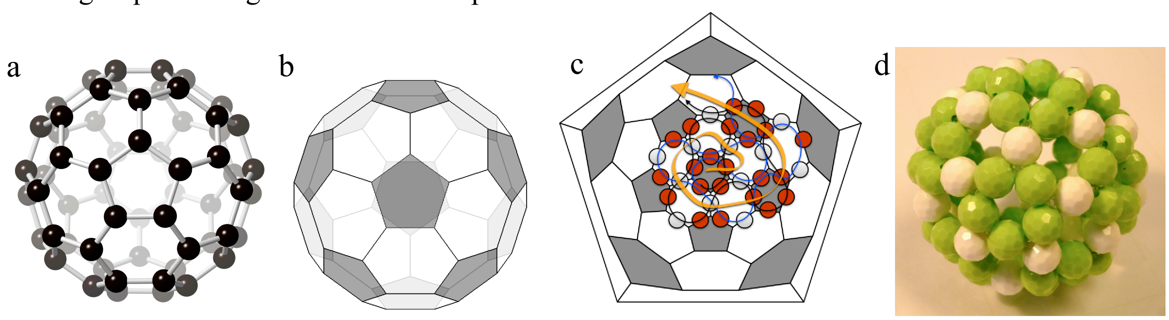


Figure 4: *Geometry, Schlegel diagram, and bead model of C_{60}.*

There is even a simpler trick to make the bead model of a C_{60} molecule, if we use beads of two different colors to represent single and double bonds respectively. This is because there are 32 polygons consisting of 12 pentagons and 20 hexagons in a C_{60}. Every pentagon is separated from neighboring pentagons by exactly one edge (Figure 4b). In the language of chemists, every pentagon consists of five single bonds, and every hexagon consists of three single and three double bonds alternatively. Therefore, we should use a single color for pentagons and two different colors alternatively for hexagons. Following this rule, we can use these colors as a mnemonic for deciding whether one needs to make a pentagon or hexagon in the next step. Another important structural feature of a C_{60} is that every pentagon is surrounded by five hexagons and every hexagon is surrounded by three pentagons and three hexagons, alternatively. If we start from a pentagon with single color and follow spiral direction carefully, eventually, one should get a bead model of C_{60} correctly without using any other information.

But to make other more complicated cage-like fullerenes, spiral codes that describe the positions of twelve pentagons along the spiral direction are very useful if they exist. For instance, the spiral code of C_{60} is [1 7 9 11 13 15 18 20 22 24 26 32]. The shape of resulting beaded structure is always similar to the shape of the corresponding microscopic fullerene. It is quite amazing that one can create the faithful structure for an arbitrary fullerene with beads so easily. A simple explanation is that hard sphere repulsion among beads effectively mimics the valence-shell electron-pair repulsion of trivalent carbon atoms in fullerene molecules.[8] A list of fullerenes up to 100 carbon atoms is given in the appendix of the book "*An Atlas of Fullerenes*" by P. W. Fowler and D. E. Manolopoulos.[9] So one can create a bead model of any fullerene easily by following its spiral code. The spiral codes of C_{70} and seven isomers of C_{80} are given as follows: C_{70}:1 [1 7 9 11 13 15 27 29 31 33 35 37]; C_{80}:1 [1 7 9 11 13 15 28 30 32 34 36 42]; C_{80}:2 [1 7 9 11 13 18 25 30 32 34 36 42]; C_{80}:3 [1 7 9 11 14 22 27 30 34 36 38 40]; C_{80}:4 [1 7 9 11 14 23 28 30 33 35 37 39]; C_{80}:5 [1 7 9 12 14 20 26 28 32 34 39 42]; C_{80}:6 [1 7 10 12 14 19 26 28 32 34 39 42]; C_{80}:7 [1 8 10 12 14 16 28 30 32 34 36 42].

In addition to their spiral codes, we give the Schlegel diagrams and possible beading directions for C_{70} and C_{80}:7 in Figures 5a and 5b.

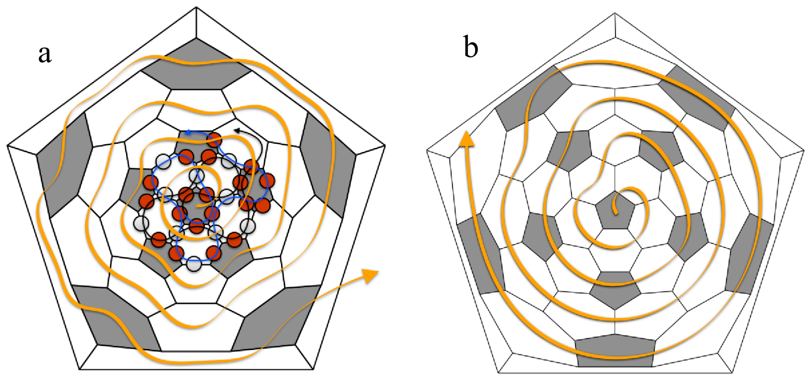


Figure 5: *Schlegel diagram of C_{70} and C_{80}:7.*

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