Visualizing Affine Regular, Area-Preserving Decompositions of Irregular 3D Pentagons and Heptagons

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Abstract

We demonstrate a simple, elegant, and visual method for decomposing irregular pentagons into a pair of affine images of the two distinct types of regular pentagons. Moreover, the sum of the area of the two affine images equals the area of the original irregular pentagon. Similarly, we decompose irregular heptagons into a triple of affine images of the three distinct types of regular heptagons. One can use these decompositions to design visually interesting sculptures reflecting these geometric relationships.

Introduction

In the 1940s, Jesse Douglas proved [2] that the median lines of irregular pentagons – even when not planar – contain the vertices of two affine regular stellar pentagons, one stellar and one non-stellar (Fig. 1). In fact, every polygon, of any degree, contains infinite families of affine regular polygons [1]. Recently, the author has discovered unique representatives of these families that preserve both location and area. We begin with virtual sculptures that reflect these mathematical relationships. We finish with geometric visualizations that show the essence of a proof of the underlying mathematics.

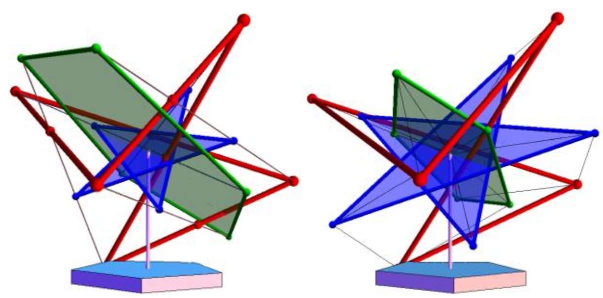


Figure 1: *Jesse Douglas' construction vs. an area-preserving construction*

Visualizing Affine Regular Decompositions of Polygons

Figure 2 shows an irregular pentagon (in thicker lines) decomposed into a large affine regular pentagon and a small affine regular stellar pentagon. Similar to distorting clip-art, a pentagon is ***affine regular*** if it is the linearly skewed, translated, rotated, and/or scaled image of a regular pentagon. In Figure 2, we also see five parallelograms connecting the corresponding vertices of each pentagon to the common centroid of the three pentagons. These parallelograms demonstrate that each vertex of the irregular pentagon is the vector sum of the vertices of the two affine regular pentagons. When measuring area we must account for folding and reflecting, which can cause the area of certain regions to count double and other regions to count as negative area. By sequentially numbering the corresponding vertices we observe that the stellar pentagon in this figure is transversed in the opposite direction to the other two pentagons. Thus, the area of the irregular pentagon equals the area of the convex pentagon minus the area of the stellar pentagon with the interior region of the stellar pentagon subtracted twice.

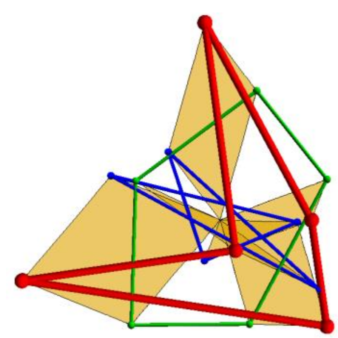


Figure 2: *Area and location relationships*

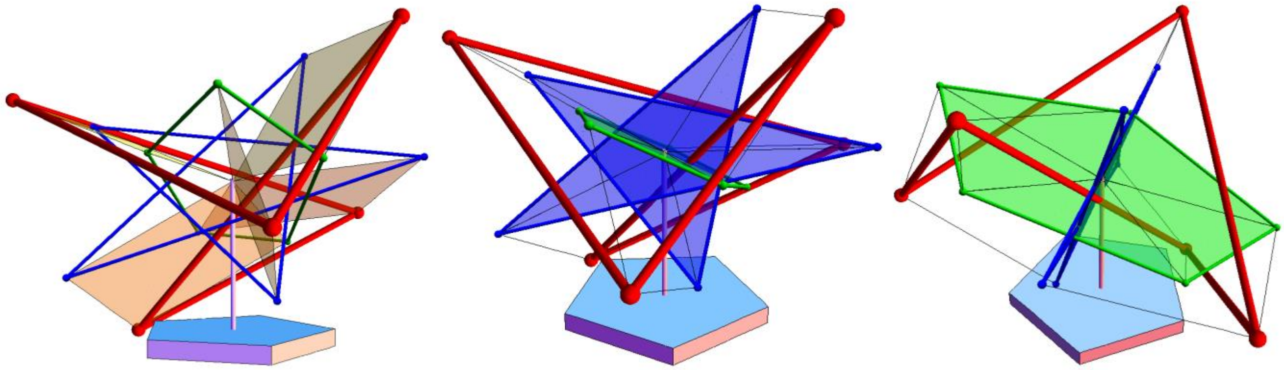


Figure 3: *Three views of the same irregular pentagon and its affine regular decomposition*

We now consider 3D computer generated sculptural visualization models. Figure 3 shows three viewpoints of two slightly different "sculptures" based upon exactly the same irregular pentagon as in Figure 1. All three figures contain the same three pentagons with wires connecting corresponding vertices. The left "sculpture" shades the five parallelograms formed by the wires to show that each vertex of this irregular pentagon is the vector sum of the corresponding vertices of the affine regular pentagons. To help see that the area is preserved under every projection, we consider these three different viewpoints with the last two oriented *almost* along edges of the two distinct affine regular pentagons. In the central figure, we see that the irregular pentagon and the stellar pentagon do indeed appear to have similar area, counting central regions twice. When oriented as shown in the top right figure we again see that areas appear to be equal. Note that the orientation of the left triangular region of the irregular pentagon is opposite and so the area must be counted as a negative. By carefully comparing the orientations of the three pentagons in the top left orientation, we see that the area of the irregular pentagon should equal the stellar minus the convex.

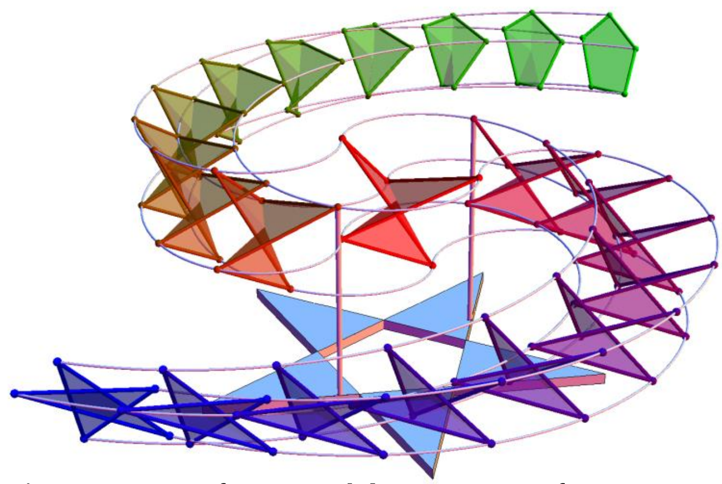


Figure 4: *A transformational decomposition of a pentagon*

The two "sculptures" to the right show the affine regular polygons folding and unfolding out from the original irregular polygon in the center. The first shows a pentagon transforming into its two affine components while the second "sculpture" shows a heptagon transforming into its three distinct affine components.

Figure 5: *A transformational decomposition of a heptagon*

Visualizing the Essence of the Mathematics

In this section we provide geometric visualizations to explain the mathematical essence that prove the relationships discussed above. We start by explaining how to use weights to create new polygons. Then we explain why we expect the shape of each of these new polygons to be affine regular. Next we verify that the original polygon is the sum of the coordinates of the vertices of these representatives. Last, we discuss area and how to verify that, when projected onto any plane, the area of the original polygon is the sum of the areas of the affine regular polygons.

Given an arbitrary heptagon $\boldsymbol{P} = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \dots, \boldsymbol{p_7}\}$, we can apply a set of *weights* $\boldsymbol{W} = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_7}\}$ to $\boldsymbol{P}$ by symmetrically calculating linear combinations of these points as follows:

$$\begin{gathered}
\boldsymbol{q_1} = \boldsymbol{w_1 p_1} + \boldsymbol{w_2 p_2} + \boldsymbol{w_3 p_3} + \boldsymbol{w_4 p_4} + \boldsymbol{w_5 p_5} + \boldsymbol{w_6 p_6} + \boldsymbol{w_7 p_7} \\
\boldsymbol{q_2} = \boldsymbol{w_1 p_2} + \boldsymbol{w_2 p_3} + \boldsymbol{w_3 p_4} + \boldsymbol{w_4 p_5} + \boldsymbol{w_5 p_6} + \boldsymbol{w_6 p_7} + \boldsymbol{w_7 p_1} \\
\vdots \\
\boldsymbol{q_7} = \boldsymbol{w_1 p_7} + \boldsymbol{w_2 p_1} + \boldsymbol{w_3 p_2} + \boldsymbol{w_4 p_3} + \boldsymbol{w_5 p_4} + \boldsymbol{w_6 p_5} + \boldsymbol{w_7 p_6}
\end{gathered}$$

to obtain a new heptagon $\boldsymbol{W}(\boldsymbol{P}) = \{\boldsymbol{q_1}, \boldsymbol{q_2}, \dots, \boldsymbol{q_7}\}$. For heptagons, we need to create representatives for each of the three distinct types of regular heptagons. Thus, we need three sets of weights. We shall use weights defined by the distance from the *y*-axis to the seven points of each of the three types of regular heptagons each inscribed in a circle of radius 2/7. The first point must lie on the *x*-axis and the order of the weights must correspond to the order of the vertices, as shown in Figure 6. We shall call these three sets of weights $\boldsymbol{W_1}, \boldsymbol{W_2}, \boldsymbol{W_3}$ respectively**.** Notice some weights are negative and each set of weights sum to zero due to their regular spacing around a circle.

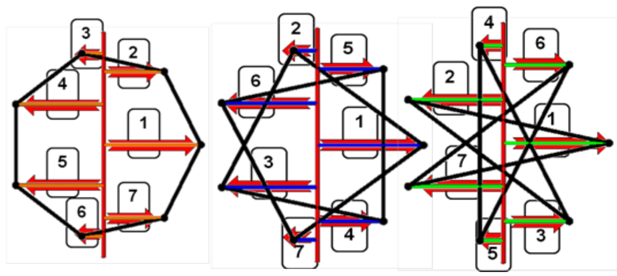


Figure 6: *Heptagon weights using regular heptagons inscribed in circles of radius 2/7*

We now wish to show that for any heptagon $P = \{p_1, p_2, \dots, p_7\}$, its image, $W_i(P)$, is always affine regular. We start with a regular heptagon $T_0 = \{r_1, r_2, \dots, r_7\}$ of the same type as W_i. We then construct a sequence of heptagons $T_1 = \{p_1, r_2, \dots, r_7\}$, $T_2 = \{p_1, p_2, r_3, \dots, r_7\}$, … , $T_7 = P$ where each heptagon is obtained by sliding the i^{th} vertex from r_i to p_i. We consider the shape of each image $W_i(T_j)$. Since T_0 is regular, then, by the symmetry of the linear combinations in the definition of $W_i(\,)$, so is $W_i(T_0)$. Let $\vec{v}$ be the vector from point r_1 to p_1, which transforms T_0 into T_1, as shown in Figure 7a. The impact of $\vec{v}$ on each point in $W_i(T_0)$ is a translation parallel to $\vec{v}$ with a magnitude defined by the weights. Since the weights are defined by the distance from a line, Figure 6, then, by similar triangles as shown in Figure 7b, $W_i(T_1)$ is an affine transformation of $W_i(T_0)$. By repeating this procedure, $W_i(P) = W_i(T_7)$ is an affine transformation of the regular heptagon $W_i(T_0)$, as shown in Figure 8.

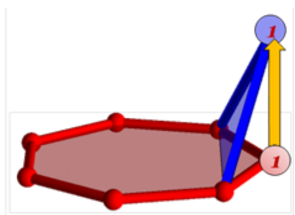


Figure 7a: T_0 & T_1

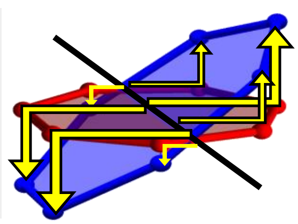


Figure 7b: $W_1(T_0)$ & $W_1(T_1)$

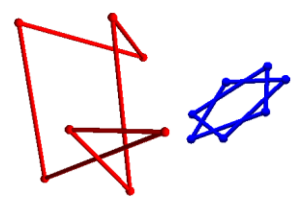


Figure 8: P & $W_2(P)$

We now verify that the vector sum of the three corresponding vertices in the affine images is a vertex in the original heptagon. Let $P = \{p_1, p_2, \dots, p_7\}$ be any heptagon centered at the origin and let W_1, W_2, and W_3 be three sets of weights defined in Figure 6. Observe that these three sets merely shuffle the same values: $W_1 = \{\frac{2}{7}, w_2, w_3, \dots, w_7\}$, $W_2 = \{\frac{2}{7}, w_3, w_5, w_7, w_2, w_4, w_6\}$, and $W_3 = \{\frac{2}{7}, w_4, w_7, w_3, w_6, w_2, w_5\}$. Let A=$W_1(P)$, B=$W_2(P)$ and C=$W_3(P)$ and let O be the heptagon with all seven vertices o_j at the centroid.

Since we centered P at the origin, O is the additive identity of heptagons and so $W_1(P)+W_2(P)+W_3(P)=A+B+C+O$. It suffices to consider the sum of the first vertex of each of these heptagons:

$$\begin{array}{rl} a_1 = & \frac{2}{7}\cdot p_1 + w_2\cdot p_2 + w_3\cdot p_3 + w_4\cdot p_4 + w_5\cdot p_5 + w_6\cdot p_6 + w_7\cdot p_7 \\ b_1 = & \frac{2}{7}\cdot p_1 + w_3\cdot p_2 + w_5\cdot p_3 + w_7\cdot p_4 + w_2\cdot p_5 + w_4\cdot p_6 + w_6\cdot p_7 \\ c_1 = & \frac{2}{7}\cdot p_1 + w_4\cdot p_2 + w_7\cdot p_3 + w_3\cdot p_4 + w_6\cdot p_5 + w_2\cdot p_6 + w_5\cdot p_7 \\ + \; o_1 = & \frac{1}{7}\cdot p_1 + \frac{1}{7}\cdot p_2 + \frac{1}{7}\cdot p_3 + \frac{1}{7}\cdot p_4 + \frac{1}{7}\cdot p_5 + \frac{1}{7}\cdot p_6 + \frac{1}{7}\cdot p_7 \\ \hline \hline & 1\cdot p_1 + 0\cdot p_2 + 0\cdot p_3 + 0\cdot p_4 + 0\cdot p_5 + 0\cdot p_6 + 0\cdot p_7 = p_1 \end{array}$$

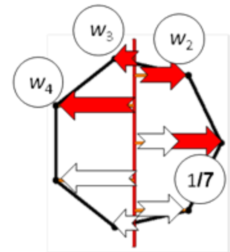
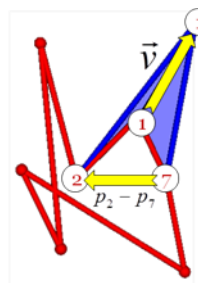


Figure 9: ½$\sum w_i = 0$

The first column is free. The pairings $w_6=w_1$, $w_5=w_2$, $w_4=w_3$ reduce the remaining columns to ½$\sum w_i$. Since the weights are equally spaced around a circle, as shown in Figure 9, this sum is zero.

We now discuss how area is preserved, that is, why Area(A) + Area(B) + Area(C) = Area(P). First we must clarify the meaning of area for non-planar polygons. Given any polygon P in R^3 with centroid O, we define area to be the sum of the area vectors of triangles Op_ip_{i+1} computed by the cross product vector $\frac{1}{2}(\overrightarrow{Op_i}\times\overrightarrow{Op_{i+1}})$. When projecting these polygons onto a plane with unit normal $\vec{n}$, the area of the projection of each of these triangles will equal plus or minus the dot product of the area vector with this unit normal vector where the sign depends upon the orientation of the triangle when projected onto the plane. The result is that triangles can have positive or negative area and overlapping regions can cancel or can double the area depending upon their orientation.



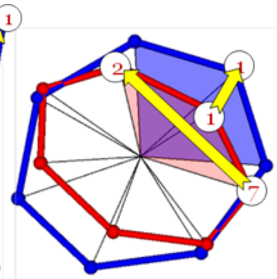


Figure 10a: T_k & T_{k+1} **Figure 10b:** $W_1(T_k)$ & $W_1(T_{+1})$

To prove that area is preserved we combine the previous methods. We use a sequence of heptagons $T_0, T_1, \ldots, T_7$ which transform the degenerate heptagon $T_0=O$ into $T_7=P$ by interchanging one vertex at a time. And we let $A_0, A_1, \ldots, A_7$ and $B_0, B_1, \ldots, B_7$ and $C_0, C_1, \ldots, C_7$ be the corresponding images under the transformations W_1, W_2 and W_3. Since, $T_0=A_0=B_0=C_0=O$ we have $Area(A_0)+Area(B_0)+Area(C_0)=0=Area(T_0)$. As before, we slide one vertex, p_1 in T_k by the vector $\vec{v}$ to produce the polygon T_{k+1}, as shown in Figure 10a. Then the increase in area, shaded in Figure 10a, is

$$Area(T_{k+1})-Area(T_k)=\tfrac{1}{2}(p_7-p_1)\times\vec{v}+\tfrac{1}{2}\vec{v}\times(p_2-p_1)=\tfrac{1}{2}\vec{v}\times(p_2-p_7).$$

To measure the increase in area between, $Area(A_k)$ and $Area(A_{k+1})$, shown in Figure 10b, we measure the increase in area of two triangles and then, using the fact that A_k and A_{k+1} are affine regular, we multiply this area by $\frac{7}{2}$. Thus,

$$Area(A_{k+1})-Area(A_k)=\tfrac{7}{2}\left[\tfrac{1}{2}(a_7+w_7\vec{v})\times(a_1+w_1\vec{v})+\tfrac{1}{2}(a_1+w_1\vec{v})\times(a_2+w_2\vec{v})\right]-$$
$$\tfrac{7}{2}\left[\tfrac{1}{2}(a_7\times a_1)+\tfrac{1}{2}(a_1\times a_2)\right]=\tfrac{7}{2}\cdot\tfrac{1}{2}\vec{v}\times[(w_7-w_2)a_1+w_1(a_2-a_7)]=\tfrac{1}{2}\vec{v}\times(a_2-a_7).$$

Basic properties of the cross product simplify the first long equation down to the second. The last formula follows from $w_7=w_2$ and $w_1=\frac{2}{7}$. The formula is the same for B and C. To complete the proof we only need to combine the A's, B's and C's to get

$$\big(Area(A_{k+1})-Area(A_k)\big)+\big(Area(B_{k+1})-Area(B_k)\big)+\big(Area(C_{k+1})-Area(C_k)\big)=$$
$$\tfrac{1}{2}\vec{v}\times[(a_2-a_7)+(b_2-b_7)+(c_2-c_7)]=\tfrac{1}{2}\vec{v}\times(p_2-p_7)=Area(T_{k+1})-Area(T_k).$$

Hence, area is preserved.

References

[1] Douglas Burkholder, *Symmetric Linear Constructions*, Bridges: Mathematical Connections in Art, Music, and Science, Conference Proceedings, Banff, 2005, pp. 403-410.

[2] Jesse Douglas, *A Theorem on Skew Pentagons*, Scripta Mathematica, Vol. 25, 1960, pp. 5-9.