Notes Relating to the Precious Properties of Regular Polygons.

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Abstract

My first reference to the property of Preciousness occured in 2003. Since then I have consistently used the term as discussed at that time. It is concerned with a sets of shapes used to create larger versions of themselves. I chose the term Precious because amongst the first contenders were two triangles that were described as Golden. I figured that if they were golden then the others were, at least, worthy of the title Precious. I am interested in Geometric and Algebraic structures that can be used to contain Artwork.

1 Introduction

Any regular polygon can be divided into a set of isosceles triangles. The investigation looks at any design using one or more of these triangles. When the design is enlarged it becomes possible to split the enlarged design into the original triangles. Rather like an embryo that, at a critical point, splits into more cells. I call the original triangles Primary Shapes, in this case, the shapes are isosceles triangles, hence I have called them Primary Isosceles Triangles. because they are isosceles and Primary means original. The enlargement factor at this critical stage I have called the Precious Ratio P. If the enlargement continues, further splitting can take place. The process can continue ad infinitum. An example of three triangles that display the property of preciousness is shown in figure 1

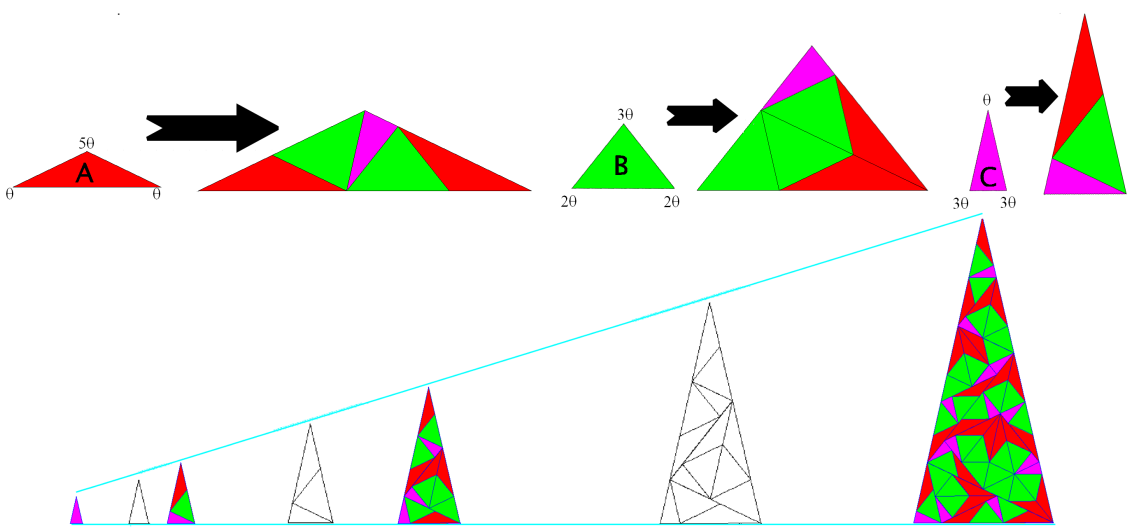

Figure 1: *A set of three isosceles triangles that can, together, be described as precious. The enlargement of triangle C can result in a sub division embryo style. FS1*

2 The property of Preciousness

If the set of isosceles triangles in figure 1 were gradually enlarged then at some point each could be split down into their primary isosceles triangles. Rather like an embryo grows and then the cells split. Furthermore, like an embryo, this division occurs over and over again.

If we take triangle C and gradually enlarge it then it can be divided into its primary triangles. Such a division can only occur when it has increased in size by a factor P, known as the Precious Ratio. For these particular triangles P = 2.246. Another sub division can occur when the enlargement is PxP. In fact, each time the enlargement is multiplied by P a sub division can take place. This can be repeated ad infinitum. See figure 1. In this example the set of shapes are isosceles triangles. I have several examples where the set of shapes are more varied. My favourites are the Tangram shapes, the Golden Tangram, and the Ostomachion as a parallelogram (unpublished). See figure 2 One of the main conditions for a set of shapes to be called Precious is for each of them to behave in a similar way to C Each member of the set must have the same Precious Ratio.

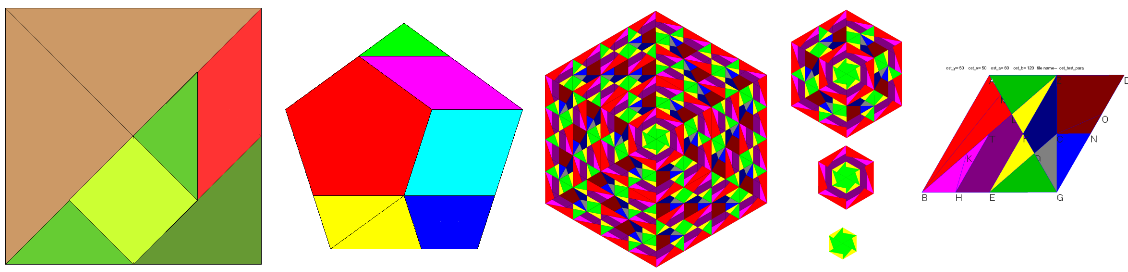

Figure 2 : *Precious sets involving shapes other than triangles. FS3*

3 The Precious Matrix and Eigen Values.

The Precious matrix for the three triangles in figure 1 tells us how many of each of the primary triangles can be used to form the enlarged version.

$$\mathbf{P} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.1}$$

A further requirement for Preciousness is that each of the Primary shapes should be involved in the design after a number of expansions. This needs to be the case even if we start with a single triangle. To verify this the nth power of the matrix (3.1) should not contain zeroes. The number of Primary shapes being n. This ensures that each primary shape will will eventually appear. This is very similar to a route matrix for a network. In this case the nth power tells us whether each of the n nodes in the network can be reached from every other node, something the network people call connected. In the case of the three triangles A, B and C the precious matrix contains no zeroes so the nth power should contain no zeroes. [6] [7] [8] [9] [10] [11]. Subsequent powers of the precious matrix shows us the number of each triangle A, B and C generated from each single triangle. For example, after 16 expansions.

$$P^{16} = \begin{pmatrix} 147494 & 183922 & 81853 \\ 183922 & 229347 & 102069 \\ 81853 & 102069 & 45425 \end{pmatrix} \tag{3.2}$$

For every primary triangle A we started with we'll get 147494 triangle A, 183922 triangle B, and so on. Now some may say, so what! The numbers become a lot more interesting when they are divided by the total of the numbers in their row.

$$\begin{pmatrix} 0.3568 & 0.445 & 0.19806 \\ 0.3568 & 0.445 & 0.19806 \\ 0.3568 & 0.445 & 0.19806 \end{pmatrix} \tag{3.3}$$

We see that the fraction of A triangles generated is always 0.3568 of the total. This is true for each of the B triangles and the C triangles. We get the same proportions whatever the original design. Of further interest there appears to be a connection between these fractions and the Precious ratio. Let the fraction of A, B and C be A_n, B_n, C_n then it seems that for these triangles that

$$B_n = P.C_n \tag{3.4}$$

$$A_n = P.C_n/(P-1) \tag{3.5}$$

These results can be verified by repeated multiplication of the Precious Matrix. They are, however, approximate and so dont constitute a proof. The Eigen Values for the Precious Matrix give some hope of logical rigour. This is beyond my current capabilities in all but simple examples. Consider the simple regular polygon with five sides in figure 3. In this case the Precious Ratio is 1.61803 the golden ratio. After 16 expansions the ratio of triangles A:B is P which is also, in this case the golden ratio. A more rigourous result could be achieved using the Eigen Value approach.

$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.6}$$

$$P^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \tag{3.7}$$

$$P^{16} = \begin{pmatrix} 3524578 & 2178309 \\ 2178309 & 1346269 \end{pmatrix} \tag{3.8}$$

The characteristic equation for a matrix is

$$(P - \lambda.I).x = 0 \tag{3.9}$$

$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.10}$$

$$\mathbf{P} = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \tag{3.11}$$

$$(2-\lambda)(1-\lambda) - 1 = 0 \tag{3.12}$$

expanding we get

$$\lambda^2 - 3\lambda + 1 = 0 \tag{3.13}$$

$$\lambda = \frac{3 \pm \sqrt{5}}{2} \tag{3.14}$$

$$\left\{ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \frac{3 \pm \sqrt{5}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.15}$$

this leads to

$$x_1 = \phi.x_2 \tag{3.16}$$

4 Criteria for a Fractal.

Some of the results from the work on Precious shapes can lead to an irregular tiling pattern filling the plane, It might lead to the formation of a fractal, It may reult in a line or a point. The Fractal Dimension is the key number in the classification of the range of designs. I looked at the Sierpinski Sieve as a starting point. In essence the expansion of a simple equilateral triangle can lead to one of the above depending on not, what you do, but more on what you leave out. Figure 4 shows the development of a Sierpinski sieve. A classic in the fractal field. Doubling the size provides an opportunity to split the large triangle into the primary triangle, embryo fashion. In this case the central triangle takes no further part. Expanding to four times the primary triangle we get another opportunity to split the triangles. Progessing with the expansion to eight times we can repeat the process, rembering that the cental triangle drops out of the process. On calculating the fractal dimension for this pattern we get 1.584 which is between 1 and 2, which I accept as a feature of a fractal. If we repeat the process, but split the central triangle as well we get a fractal dimension of 2 i.e a plane tiling that will, eventually, fill the plane, see figure 4. Leaving out the top triangle leads to a fractal with the same fractal dimension. Leaving out two of the triangles leads to a fractal dimension of 1. i.e. a line. Leaving out 3 triangles leads to a fractal dimension of 0, i.e. a point. When I create a fractal from a set of precious shapes I use a similar arguments to classify my designs. I have in mind to look at whether I can find a connection between the fractal dimension and the Precious Ratio, but not today! Figure 5 shows the development of a simple fractal using the scheme in figure 1

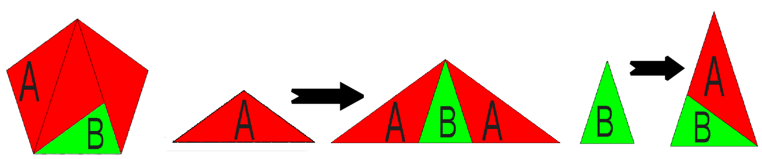

Figure 3: *Primary Isosceles Triangles for the regular Pentagon. FS2A*

5 Basic Geometry of the Regular Polygon

Bearing in mind that the vertices of a regular polygon lie on a circle, and, because it is regular, anything said about one vertex is true for any other vertex. With reference to Figure 6. For a regular polygon with n sides, the cental angle is 2θ where $\theta = 180/n$. The angle between any vertex and the two ends of any side is always θ. Consequently, the angle between any vertex and k adjacent sides is always $k\theta$. If we join B to E forming triangle OBE the the angles of the triangle are 2θ, 3θ and 8θ. The sum of which is $n\theta$. For any triangle, so formed, the sum would always be $n\theta$. Archimedes was interested in triangles with their angles forming a geomtric progression. For instance, a triangle whose angles are in the ratio 1:2:4 would require a polygon with 1+2+4=7 sides. A ratio of 1:3:9 would fit on a 13-gon, and a 3:4:5 would need a 12:gon. An

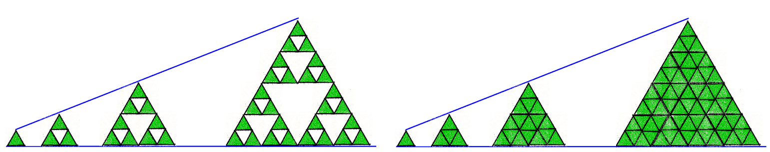

Figure 4: *This shows the development of a Sierpinsk sieve. Essentially the enlarged triangle divides into four. The cenral triangle then takes no further part in the process. This leads to a fractal with a dimension of 1.584 i.e between 1 and 2. Continuing with the central triangle leads to a fractal dimension of 2, i.e. a plane tiling. FS9*

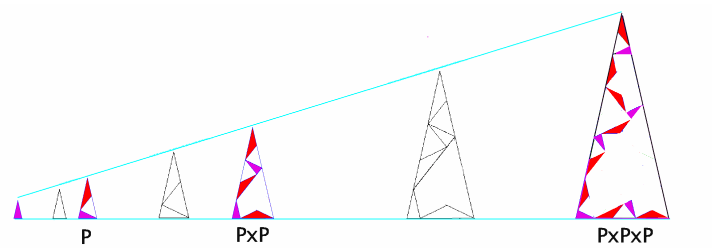

Figure 5 : *The development of a fractal using the scheme shown in figure 1 FS101*

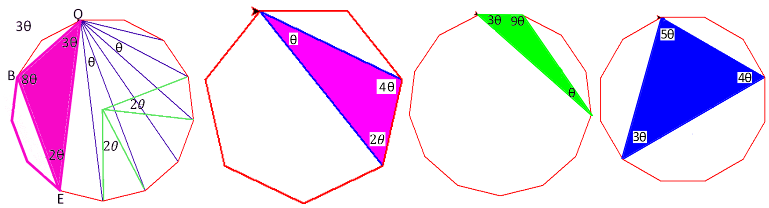

Figure 6 : *Some basic geometry of a regular polygon FS102*

80-gon would be needed to celebtate Newton's birthday 25:12:43! Some of these can be seen in figure 6.

With reference to figure 7 and Using the previous results

$$HAI = \theta \tag{5.1}$$

since IG'H is external to HG'A we get

$$HG'I = 2\theta \tag{5.2}$$

now since $HI = GH$

$$HG'I = 2\theta \tag{5.3}$$

similarly angle$IAJ = \theta$ and so on.... If HGA is reflected in HA we get

$$HG = HG' = IH' \tag{5.4}$$

So, triangle IHG' is isosceles and IH'J=2θ since IH'J is external to IH'A Repeating this process we can fill the whole polygon with isosceles triangles. These triangles I have referred to as the Primary Isosceles Triangles. Figure 7 shows the result for a 30-gon.

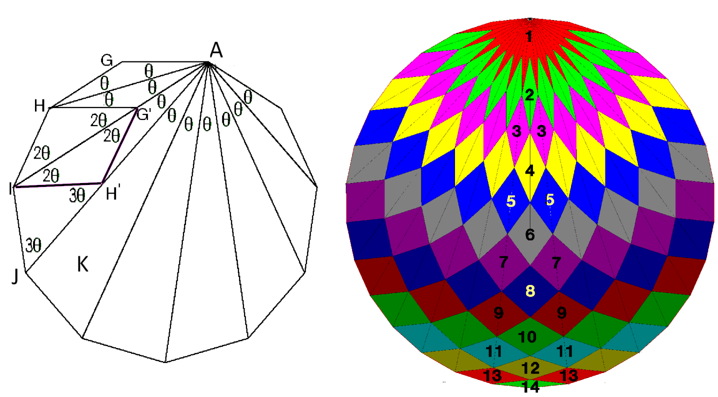

Figure 7 : *Filling a polygon with its Primary Isosceles Triangles FS103*

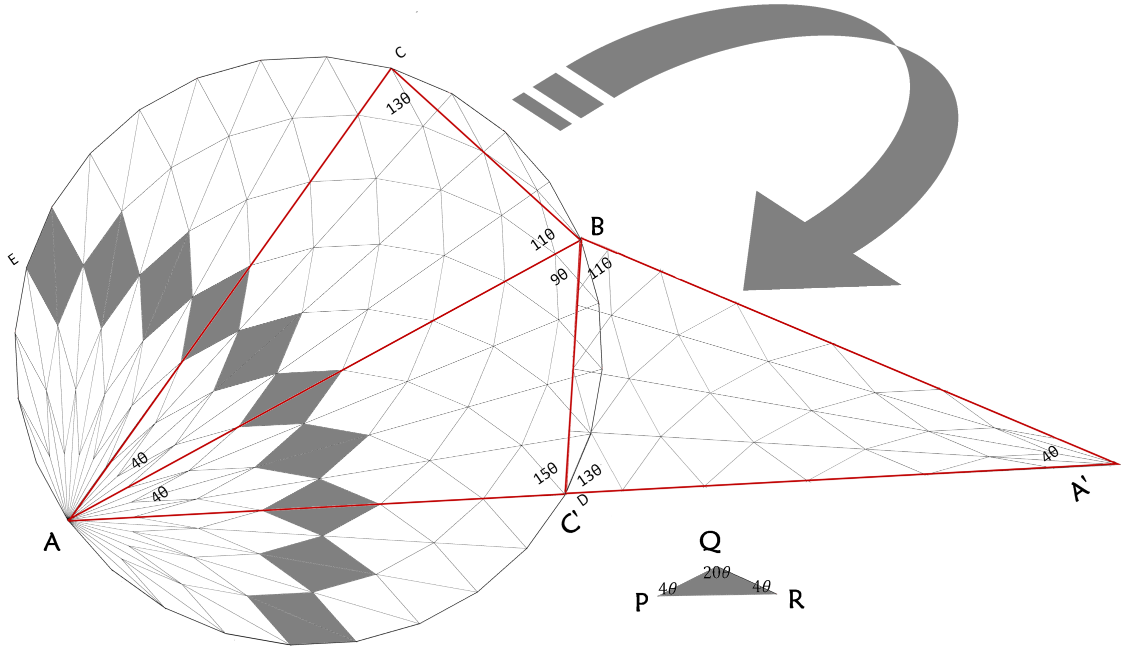

Figure 8 : *Constructing enlarged isosceles triangles Prec01*

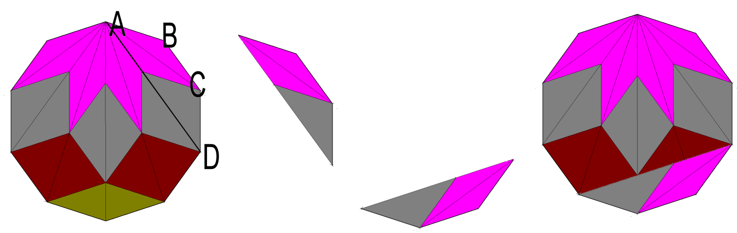

Figure 9 : *Dividing a polygon along a diagonal FS104*

6 Searching for the enlarged Isosceles Triangles.

For a Precious relationship to exist we need, for each of the primary isosceles triangles a large triangle made from some, or all, of the primary isosceles triangles. As a starting point I have identified a set of triangles that are similar to the primary isoscrles triangles each having the same enlargement factor. In addition, each large triangle has an area equal to an integral sum of primary isosceles triangles. With reference to figure 8, I have drawn two triangles along a diagonal of the polygon. The angles are 4θ, 13θ and 11θ for one of the triangles and 4θ, 15θ and 9θ. Using the same diagonal AB I could have drawn triangle with CAB, and CDB equal to θ, 2θ and so on up to 13θ. From earlier arguments the area of each will be the sum of an integral number of primary isosceles triangles. Rotate triangleABC about B so that CB sits on DB forming an isosceles triangle similar to PQR coloured mid grey. Noting that the angles 15θ and 13θ form a straight line because 13 + 15 is equal to n, 28 in this case. From the geometry we get

$$AB = 1 + 2(\cos(2\theta) + \cos(4\theta) + \cos(6\theta) + \cos(8\theta) + \cos(10\theta) + \cos(12\theta))AB = 8.8752454 \quad (6.1)$$

If we assume that the polygon has a unit side then the precious ratio is

$$AB/1 = 8.8752454 \quad (6.2)$$

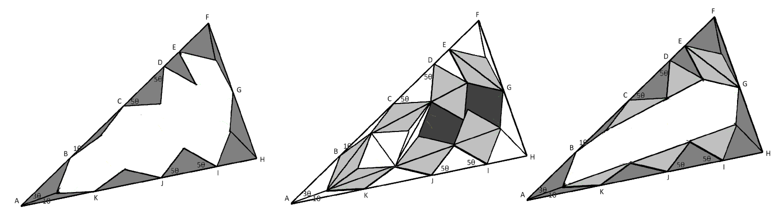

Figure 10 : *Fitting Primary Isosceles Triangles around the border Vec05*

The previous arguments could be repeated for any polygon. This leaves the not insignificant problem of actually fitting the primary triangles into its larger version.

7 Fitting the Primary Isosceles Triangles around the border using a vector approach.

A vector is a line that has both magnitude and direction. An isosceles triangle can be thought of as being two unit vectors with an angle between them. The resultant is the third side. In figure 10 the triangle AFH has an area equal to an integral number of primary isosceles triangles from the 22-gon. In addition, the lengths of the sides are made up from various combinations of sides of triangles. The triangle AFH can be described as a series of vectors which I can state as a string that can be interpreted by a piece of software.

//rt3//fds1/fds3/fds5/fds7/fds9//rt14//fds1/fds3//rt12//fds1/fds3/fds5/fds7//rt18//

The term rt3 means that the next vector is at an angle 3θ to the right of the current direction. where for this example $\theta = 180/22$. Terms like fds3 is a vector of length $2(\cos(3\theta))$ in the current direction. The different operators are separated by an / some are grouped together between a pair of //, These groups indicate where the order of the vectors is uncertain. The first such group is a single operation //rt3// right 3θ , so no issue there, the next group is //fds1/fds3/fds5/fds7/fds9//. There is uncertaintiy about the order of these operations. This is where combinatorics comes in. There are 5 vectors in the list so we choose a number 1,2,3,4 or 5 and take that member of the list as the first vector and then remove it from the list leaving 4 vectors remaining. Choosing a number from 1,2,3 and 4 gives us the next vector... and so on until there is 1 vector left, the last one. A sequence of numbers 21121 would result in //fds3/fds1/fds5/fds9/fds7// see figure 10. Continuing around the triangle we can produce a valid border, although not necessarily the one we want. I have three ways of generating the number sequence. The first is to start with the lowest number, 11111 and do a form of addition until getting to the largest number 54321. The addition is unusual in that each column maximum is one larger than the column on its right. The second method is to start with the lagest 54321 and repeatedly subtract 1 until you get to 11111. The third way is to generate 5 random numbers, one for each column. Each method allows the use of more than one computer each having a different start point.

8 Filling the centre of the triangle.

The software then attempts to fill the central part with pairs of triangles, leaving unit edges. and saves a thumbnail version of its result. Often some straight edges are evident which can be considered as a new but simpler, shape. Figure 11 shows how a number of different configurations were genereated to complete the jig saw. I have, managed to create sets of Precious triangle for most, if not all, the regular polygons, up to the 30-gon see Figure 12

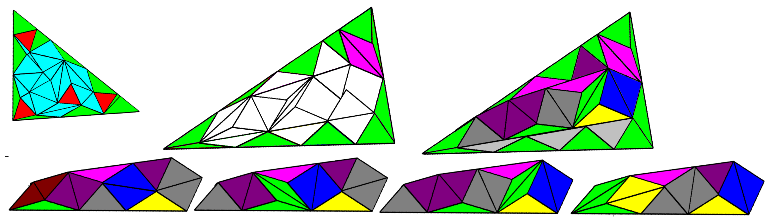

Figure 11 : *Fitting Primary Isosceles Triangles around the border Comb03B*

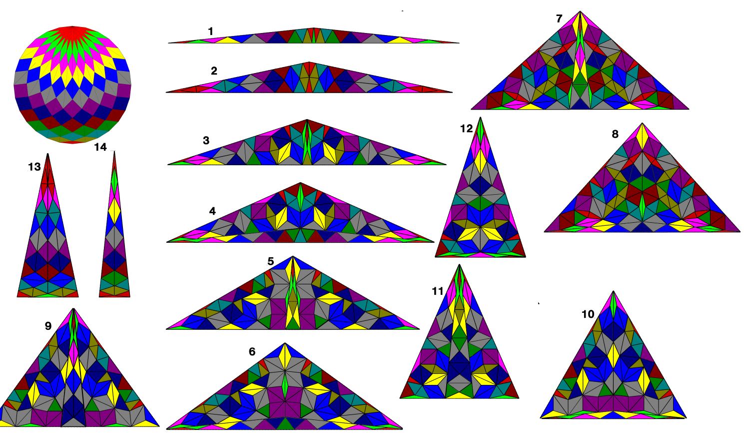

Figure 12 : *One set of enlarged triangles, for the 30-gon. These are filled with the Primary Isosceles triangles, displaying mirror symmetry. poly30*

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