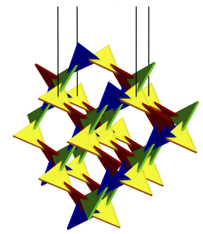
Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture

Folded Strips of Rhombuses, and a Plea for the $\sqrt{2} : 1$ Rhombus

Tom Verhoeff *
Department of Mathematics and Computer Science
Eindhoven University of Technology
Den Dolech 2
5612 AZ Eindhoven, Netherlands
T.Verhoeff@tue.nl

Koos Verhoeff
Valkenswaard, Netherlands



Abstract

One way to construct, from filled polygons, a hollow beam with polygonal cross section is to make prismatic sections from squares or rectangles and attach these back-to-back. In this paper, we explore an alternative way, based on folding a single strip of rhombuses into a discrete helix. By taking rhombuses with an appropriate aspect ratio, you can control the cross section of the resulting beam.

Using a rhombus with an aspect ratio of $\sqrt{2} : 1$ yields a triangular beam. This rhombus turns out to be a particularly fruitful construction element (alas, discontinued by Polydron). Triangular beams of this kind can be connected at an angle, in various ways, without cutting rhombuses. The resulting joints are regular miter joints, or false miter joints.

We provide a mathematical analysis and show some elegant shapes constructed from such triangular rhombus-based beams. One of these shapes is a doubly-linked octagon. Another shape is a trefoil knot, which can be linked into an interesting space-spanning structure known as triamond, and that led to the *Bamboozle*.

1 Introduction

Figure 1 (left) illustrates how one can construct from square panels a hollow beam (tube) with a pentagonal cross section. The beam in the figure consists of three sections connected one after the other, each section being an open prism with five squares as walls. You can also view this as five flat strips of three squares that form the five longitudinal faces of a pentagonal beam (an elongated prism).

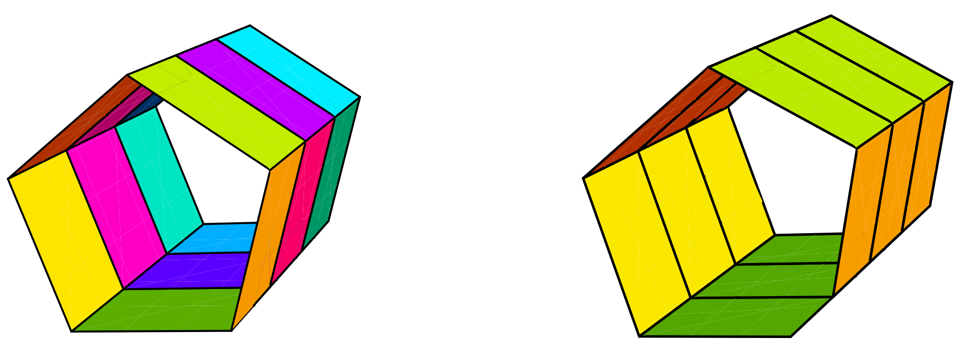


Figure 1 : *Beam with pentagonal cross section, made from prismatic sections of five squares (left), and made from one strip of rhombuses folded into a helix (right); look carefully for the difference*

On the right in Figure 1, you see an alternative construction involving rhombuses that follow a discrete helix. In Figure 2, a strip is shown as it is rolled tightly into a pentagonal beam part.

In Section 2, we analyze this construction further. Section 3 investigates constructions with triangular beams of this kind, in particular, several ways of joining two such beams without cutting the rhombuses. Various shapes and intriguing artworks are shown in Sections 4 and 5. Section 6 concludes the article.

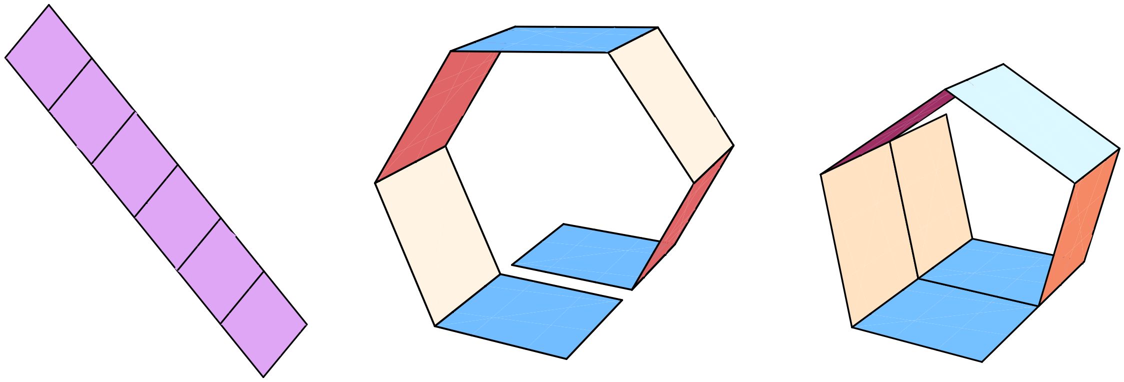


Figure 2 : *Strip of rhombuses, lying flat (left), loosely folded (middle), tightly folded (right))*

2 Mathematical Analysis

Let us study some mathematical aspects of a strip of rhombuses as it is folded into a beam. We assume, without loss of generality, that each rhombus in the strip has side length 1. The acute angle of the rhombus is α, with $0 < \alpha \leq 90°$. The aspect ratio of the rhombus, that is, the ratio of the lengths of its diagonals (long to short), is $1 : \tan(\alpha/2) = \cot(\alpha/2) : 1$.

The strip is folded such that each rhombus is rotated by an angle ϕ about the edge shared with the preceding rhombus in the strip, where $\phi = 0$ corresponds to no folding. The interior angle between two adjacent rhombuses is then $180° - \phi$. For the pentagonal cross section we need $\phi = 360°/5$.

The folded strip forms a discrete helix. In this helix, every rhombus steps a distance of $\cos\alpha$ forward along the helix axis. Thus, to fold the strip tightly into a beam with a regular n-gon as cross section, we need to have $\cos\alpha = 1/n$, so that n steps move forward along the axis by one side length. This rhombus has an aspect ratio $a : 1$ with

$$a = \cot\left(\frac{1}{2}\arccos\frac{1}{n}\right) = \sqrt{\frac{n+1}{n-1}} \tag{1}$$

Table 1 lists the parameters for triangular through octagonal cross sections.

n	3	4	5	6	7	8
a	$\sqrt{2}$	$\sqrt{\frac{5}{3}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{7}{5}}$	$\frac{2}{\sqrt{3}}$	$\frac{3}{\sqrt{7}}$
	1.41421	1.29099	1.22474	1.18322	1.1547	1.13389
α	70.5288	75.5225	78.463	80.4059	81.7868	82.8192

Table 1 : *Rhombus aspect ratio $a : 1$ and angle α to obtain various regular n-gons as cross section*

3 The Triangular Beam from $\sqrt{2} : 1$ Rhombuses and Its Joints

When constructing triangular beams from $\sqrt{2} : 1$ rhombuses as described in the preceding section, you soon discover that they can be connected at an angle (we did so with Polydron [3]). This only appears to work for triangular beams, so we restrict ourselves to those. Figure 3 shows four ways[1] to connect two triangular beams from rhombuses. The top-left connection (a) forms a straight angle, and is not interesting

[1]Other joints are possible when admitting *half rhombuses*, but we will not pursue that here.

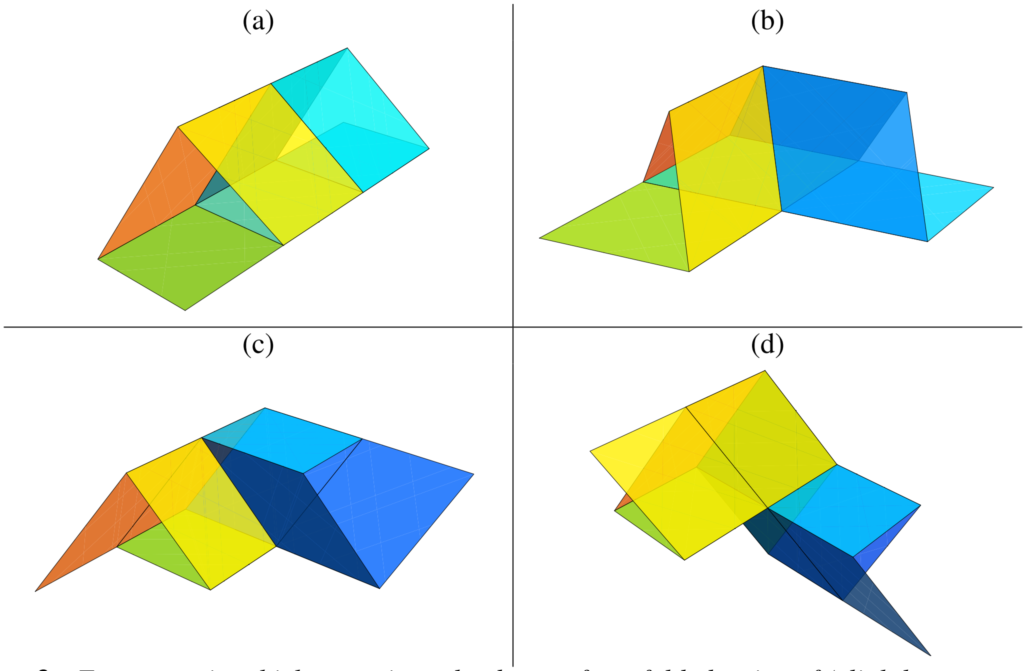


Figure 3 : *Four ways in which two triangular beams from folded strips of (slightly transparent) rhombuses, can meet, to form a continuous beam*

for constructing shapes other than a line. To the right of it, there is a joint (b) at $109.5°$ (the obtuse angle of the rhombus). Below these, there are two joints (c) and (d) both at $70.5°$ (the acute angle of the rhombus).

A bit further analysis reveals that joint (b) is in fact a *regular miter joint* [5, 6], where the two meeting helices have opposite handedness. Note that in some sense the *connecting rhombus* belongs to both beams, though it may be better to attribute half a rhombus to each beam, as dictated by the miter joint's cut plane.

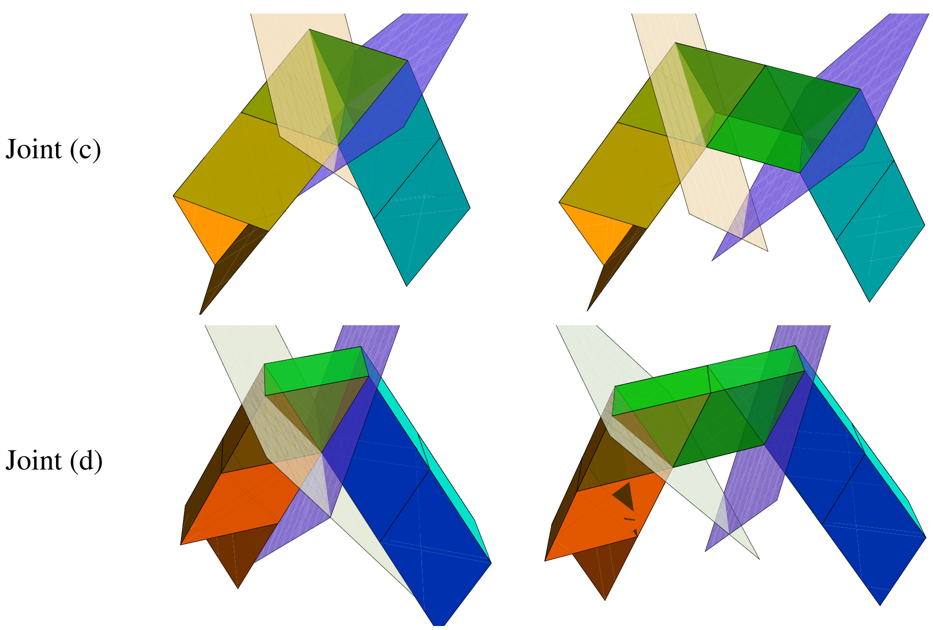


Figure 4 : *The $70.5°$ joints viewed as two consecutive regular miter joints with short middle beam (cut planes shown); right: versions with longer middle beam*

The joints (c) and (d) clearly are neither regular nor skew miter joints [5], since there is no single cut plane to yield two beveled triangular beams. Note that the meeting helices have the same handedness. They could be considered *false miter joints*, that is, instead of . Moreover, for joint (c), you can see that two pairs of beam faces connect by a *regular fold joint* [5], with the two folds lying in the exterior bisector plane of the angle (also see Fig. 4, top left); but the third pair of beam faces then do not meet. However, it is better to view them as two consecutive regular miter joint with a short middle beam, as illustrated in Figure 4. The middle beam is so short that one of its side faces (joint (c)) or edges (joint (d)) disappears. The figure also shows versions with a longer middle beam. The middle beam is transparent.

The *roll angle* [8] between two consecutive joints is the angle between the two planes that span adjacent joint angles. In both cases (c) and (d), the roll angle is $\pm 120°$, because the beam's cross section is an equilateral triangle. Hence, the result of connecting these triangular beams can be a *constant-torsion path* in the sense of [7]. Since the triangular cross section has the $120°$ rotation as symmetry, a closed path of such beams will always be *properly closed* [5, 8], that is, the longitudinal edges also meet at the closure.

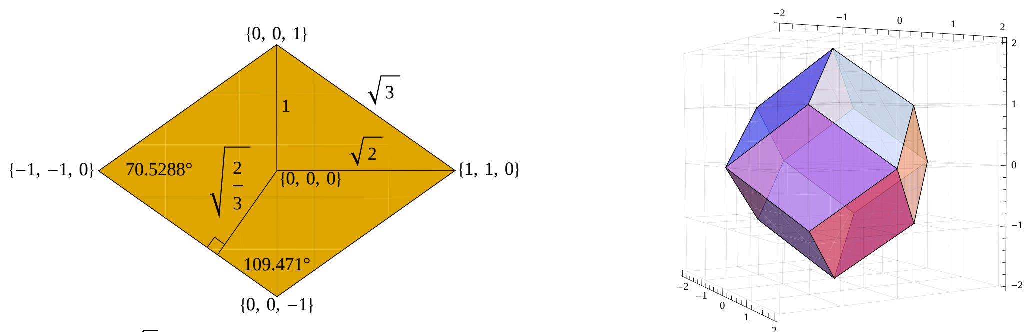


Figure 5 : *The* $\sqrt{2}:1$ *rhombus, with integer coordinates in the 3D grid, and some characteristic numbers (left); a rhombic dodecahdron on lattice points (right)*

As a consequence of the miter angle of $109,5°$ and the roll angle of $120°$, the beam directions in shapes made by joining these triangular beams correspond to the four main diagonals of the cube. The rhombuses appear in the twelve orientations of the *rhombic dodecahedron* (see Fig. 5), which consists of twelve $\sqrt{2}:1$ rhombuses. Hence, all rhombus vertices in such triangular beams can have integer coordinates. The triangular beam traverses a path in (a sublattice of) the body-centered cubic (bcc) lattice.

4 Closed Shapes from Triangular Beams

Figure 6 shows some simple closed shapes that can be made from triangular beams of $\sqrt{2}:1$ rhombuses. The hexagon side lengths can be varied independently in three directions. The length of a beam is best measured in terms of the total number of rhombuses involved; both end rhombuses are shared and count for one half. The hexagons shown have side lengths (4, 4, 4), (4, 4, 1), (1, 1, 1), (5, 5, 5), (5, 5, 2), and (2, 2, 2) respectively. The two smaller ones have a line as opening in the middle! All are constant-torsion paths.

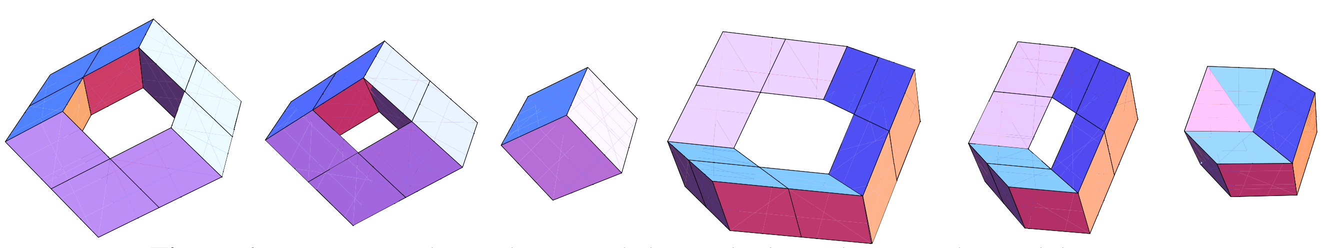
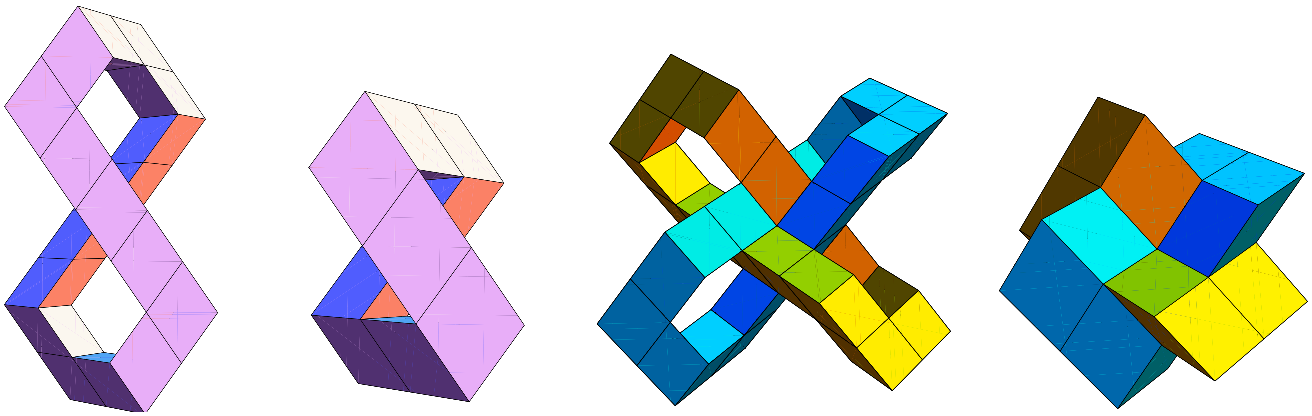


Figure 6 : *Some (non-planar) hexagonal shapes, both regular, irregular, and degenerate*



(a) (b) (c) (d)

Figure 7 : *Eight-shaped octagon; (a) loose; (b): tight; (c) and (d): doubly-linked pairs*

In Figure 7 (a) and (b), two sizes of eight-shaped octagons are shown. These are not constant-torsion paths, since there are adjacent angle pairs that are co-planar (torsion 0). Two copies can be linked to obtain a nicely symmetric link (linking number two), also known as Solomon's knot, or L4a1, see Fig. 7 (c) and (d). The shape (d) turns out to be a *space filler*. Again, the size of these shapes can be varied, but for the octagons this needs some care. The four directions cannot be varied independently. To obtain a closed path, the total displacement vector along the path must add up to zero. We can decompose that vector along the four beam directions. Let us call these direction vectors v_i for $i = 1, 2, 3, 4$, pointing towards the vertices of a tetrahedron, such that $v_1+v_2+v_3+v_4 = 0$. Let a_i be the accumulated signed length of beams in direction v_i. For closure we need $\sum_{i=1}^{4} a_i v_i = 0$. An obvious solution is to have all $a_i = 0$, as is the case for the hexagons. We call such a shape *balanced*. However, there is (only) one other solution: $a_1 = a_2 = a_3 = a_4 = a \neq 0$, which we call *unbalanced*. The eight-shaped octagons are unbalanced. For instance, the octagon on the left has beam lengths $12v_1 - 4v_2 + 4v_4 - 4v_1 + 12v_2 - 4v_1 + 4v_3 - 4v_2 = 4\sum v_i$.

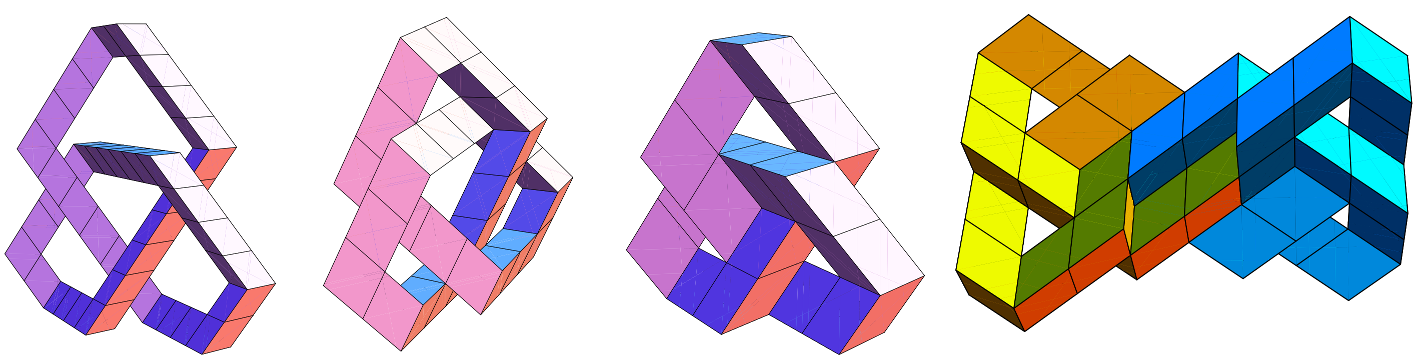


Figure 8 : *Symmetric trefoil knots; left: loose; middle two: tight; right: snugly linked pair*

Figure 8 shows some symmetric trefoil knots, both loose and tight. They all exhibit the characteristic order 2 and order 3 symmetries of the trefoil. It turns out that the tight versions can be snugly linked. This linkage is explored further in the next section.

5 Artwork

In this section, we describe some shapes from triangular beams of $\sqrt{2}$: 1 rhombuses that were actually constructed as artworks, or from which artwork was inspired. The rhombuses themselves, however, are no longer visible. Figure 9 shows a wooden doubly-linked pair of tight octagons. The two octagons were actually constructed as quadrangles using false miter joints. All the eight beam segments involved are identical (the figure also shows one loose beam segment).

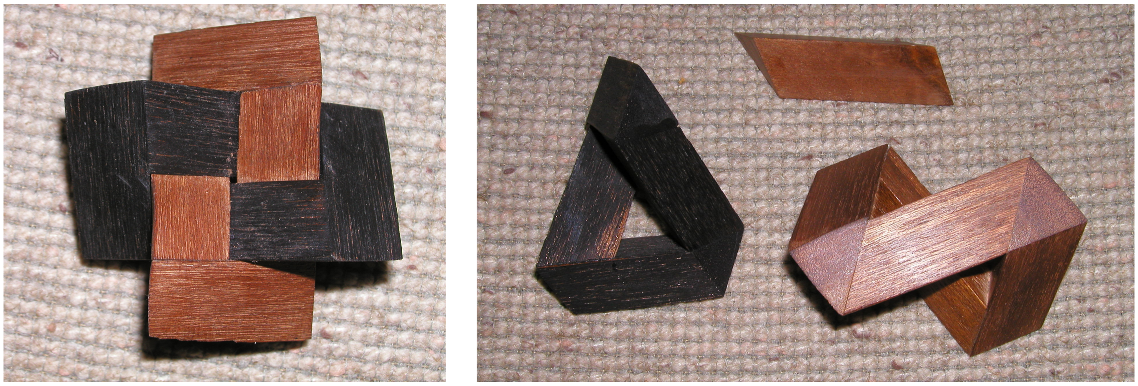


Figure 9 : *Doubly-linked pair of octagons in light and dark wood (left); decomposed into two (non-planar) quadrangles with false miter joints, and one loose beam (right)*

Two wooden trefoil knots are shown in Figure 10. Trefoil knots have an intrinsic beauty, which is enhanced by the solid wooden look and feel. As observed in the preceding section, two tight trefoil knots can be snugly linked. An obvious question is: What happens if all three loops of a trefoil knot are so filled by linking in trefoils, and then fill in the loops of those new trefoils, etc.? The number of trefoils will grow exponentially, leading either to a self-intersecting chaos, or to a nicely repeating pattern, where 'newly' added generations of trefoils coincide with prior trefoils. We had no clue when we started to investigate this.

Figure 10 : *Trefoil knots in wood; left: loose; right: tight*

To simplify the rendering, we abstracted each trefoil to a triangle. Two linked trefoils (Fig. 8, right) meet at $70.5°$ ($\arccos 1/3$), the acute angle of the $\sqrt{2} : 1$ rhombus. Figure 11 shows what happens after a few generations. The color of a triangle is determined by its normal vector; these turn out to point in only four directions. To our surprise, triangles from different branches meet only in the fifth generation. In fact, they nicely meet in exactly the right positions to give rise to a regular pattern, rather than explode into self-intersecting exponential chaos. The smallest cycle in this structure consists of ten triangles (knots).

It took us a while to determine the structure and its symmetries. In hindsight, it may seem easy. Figure 11 (right) provides some further insight by replacing the triangles with cubes. The structure appears to be both little known, and extremely beautiful. It was already described as early as 1933, but even recently triggered discussion [2, 4]. George Hart has devoted a webpage to it [1]. It is known by various names: Laves's graph of girth ten, (10,3)-a, srs, K4 crystal, and triamond. The triamond structure is uniquely determined as number 214 among the 230 space groups by the properties of being *chiral* (not mirror-symmetric) and *strongly isotropic* (everywhere the same). The only other strongly isotropic 3D structure is diamond, which is mirror symmetric (achiral).

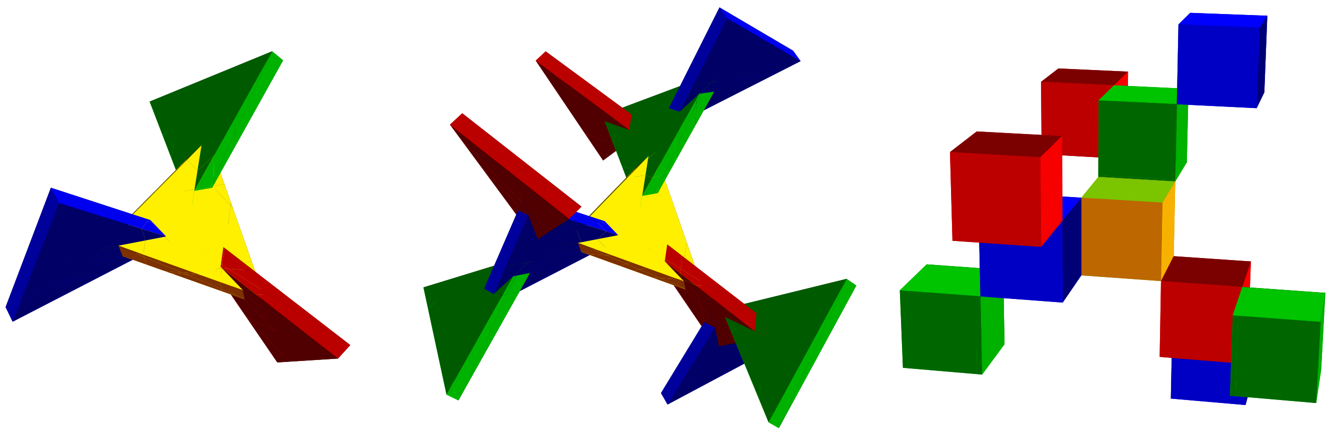


Figure 11 : *Triangles intersecting at* $70.5°$*: one generation added (left); two (middle); as cubes (right)*

A large version (see Fig. 12), consisting of 51 triangles, intersecting in the triamond pattern, executed in polished red, green, blue, and yellow acrylic was recently installed in the MetaForum building of the Department of Mathematics & Computer Science at Eindhoven University of Technology. We named it *Bamboozle* [11].

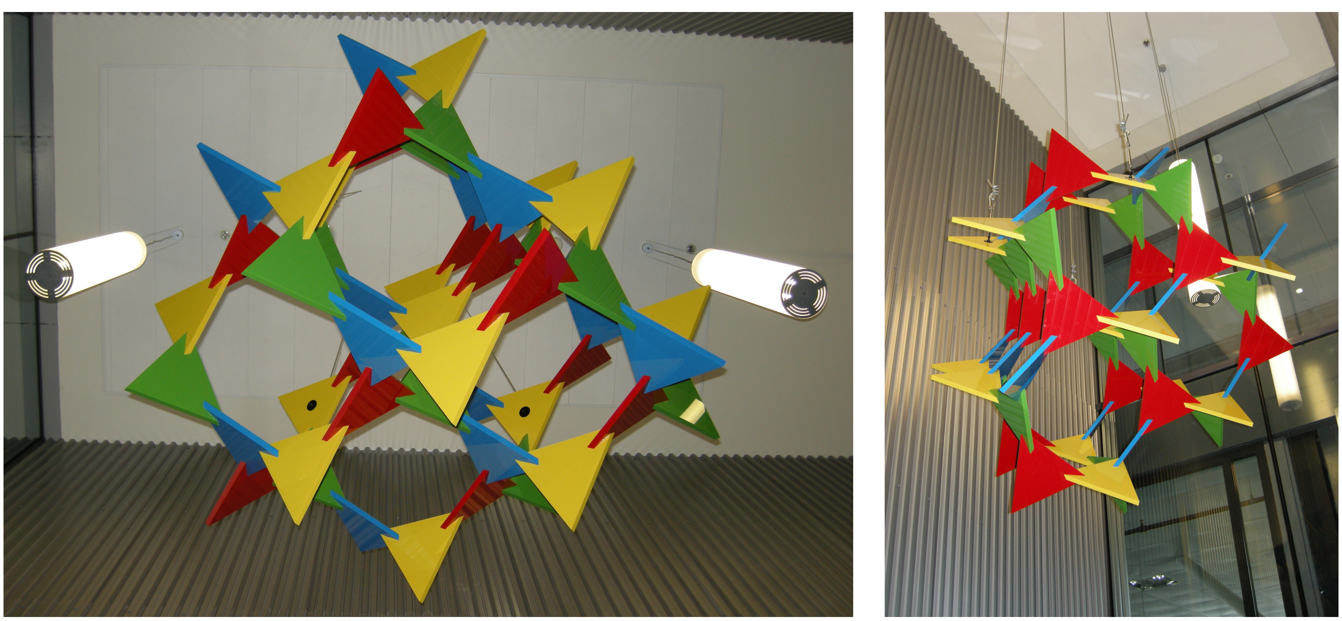


Figure 12 : *Bamboozle in MetaForum at Eindhoven University of Technology*

6 Conclusion

Polydron [3] used to offer a $\sqrt{2}:1$ rhombus, but unfortunately it was discontinued. One reason could be that it was thought to be useful only for constructing the (not so exciting) rhombic dodecahedron. The golden rhombus, with the golden ratio $(1+\sqrt{5})/2$ as aspect ratio, is still available in Polydron. The golden rhombus appears in the (admittedly, quite elegant) rhombic triacontahedron (30) and rhombic hexecontahedron (60), but does not have much more use[2]. We have shown that the $\sqrt{2}:1$ rhombus, however, is a versatile construction element for triangular beams that can be joined at angles of approximately $70.5°$ and $109.5°$, resulting in paths with roll angles of 0 and $\pm 120°$. Using this $\sqrt{2}:1$ rhombus, various intriguing constructions were realized. In particular, it led to the rediscovery of the triamond structure, and to the *Bamboozle* artwork.

Constructions involving half rhombuses deserve further investigation. Also, it could be interesting to look into joints where three or more triangular beams meet [9]. Finally, there are two possible connections to the helix structures of [10]; on one hand, the beams that we studied here are helices of rhombuses; on the other hand, using these beams, one can again construct nice helices that can be elegantly intertwined.

Acknowledgments Walt Ballegooijen pointed out to us that the doubly-linked tight octagons are in fact a space filler. Walt also helped us recognize the (10,3)-a structure in *Bamboozle*.

References

[1] G. Hart. *The (10,3)-a Network*. `http://www.georgehart.com/rp/10-3.html` (accessed 24 Jan. 2013).

[2] S.T. Hyde, M. O'Keeffe, and D.M. Proserpio. "A Short History of an Elusive Yet Ubiquitous Structure in Chemistry, Materials, and Mathematics", *Angewandte Chemie Int. Edition*, **47**:7996–8000 (2008).

[3] *Polydron*, a geometric construction product. `http://www.polydron.co.uk` (accessed 24 Jan. 2013).

[4] T. Sunada. "Crystals That Nature Might Miss Creating", *Notices of the AMS*, **55**(2):208–215 (Feb. 2008).

[5] T. Verhoeff, K. Verhoeff. "The Mathematics of Mitering and Its Artful Application", *Bridges Leeuwarden: Mathematical Connections in Art, Music, and Science, Proceedings of the Eleventh Annual Bridges Conference, in The Netherlands*, pp. 225–234, July 2008.

[6] T. Verhoeff. "Miter Joint and Fold Joint". From *The Wolfram Demonstrations Project*, `http://demonstrations.wolfram.com/MiterJointAndFoldJoint` (accessed 24 Jan. 2013).

[7] T. Verhoeff, K. Verhoeff. "Regular 3D Polygonal Circuits of Constant Torsion", *Bridges Banff: Mathematics, Music, Art, Architecture, Culture, Proceedings of the Twelfth Annual Bridges Conference, in Canada*, pp.223–230, July 2009.

[8] T. Verhoeff. "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry". *Int. J. of Arts and Technology*, **3**(2/3):288–319 (2010).

[9] T. Verhoeff, K. Verhoeff. "Branching Miter Joints: Principles and Artwork". In: George W. Hart, Reza Sarhangi (Eds.), *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture*. Tessellations Publishing, pp.27–34, July 2010.

[10] T. Verhoeff, K. Verhoeff. "From Chain-link Fence to Space-spanning Mathematical Structures". In: Reza Sarhangi and Carlo Séquin (Eds.), *Proceedings of Bridges 2011: Mathematics, Music, Art, Architecture, Culture*. Tessellations Publishing, pp.73–80, July 2011.

[11] T. Verhoeff. *Bamboozle*. `http://www.win.tue.nl/bamboozle` (accessed 24 Jan. 2013).

[2] Exercise: What do you get when you tightly fold a strip of golden rhombuses?