OrbiFold and Cut

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Abstract

People have been unknowingly folding and cutting orbifolds of symmetry groups for hundreds of years, in the form of paper snowflakes and paper dolls. But now that more research has been done on orbifolds, we know of more foldable symmetry groups than ever before. Previously, people have explored how to fold paper into the orbifolds of the Frieze groups and 2D point groups, but in this workshop we'll take our folding skills to the limit with wallpaper and spherical groups. Using large sheets of paper, spherical beach balls, and some sharp craft knives, we'll see if we can make them all!

Workshop Activity

In this workshop, participants will learn how to fold and cut beach balls to make objects with spherical symmetry, in a way analogous to how one would fold and cut a flat sheet of paper to create a symmetric snowflake. Along the way, we'll explore all the possible symmetry groups on the plane and sphere, how the planar point groups connect to prime numbers and combinatorics, and even think about how one might theoretically fold more difficult groups. The possible applications in the classroom range from a simple hands-on introduction to symmetry suitable for anyone old enough to handle scissors, to an interactive conceptual aid when studying group theory, spherical geometry, orbifolds, or topics at the "cutting edge" of symmetry research!

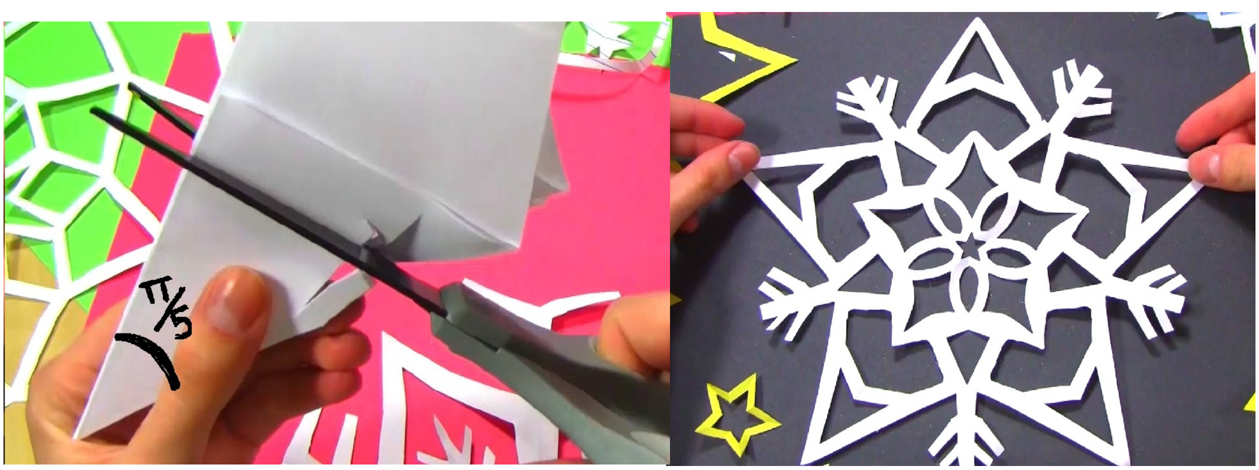

Figure 1: *Cutting and opening a 5-fold symmetric snowflake.*

Materials

- Paper
- Beachballs (many colors, 2 per sphereflake [a structural beach ball will be inflated inside the cut one])
- Scissors and craft knives
- Whiteboard markers (for erasable drawing on beach balls)
- Protractor (helpful when attempting to fold higher primes)
- Glue/tape (to attach complicated designs to inner beach ball. Use a flexible glue like rubber cement)
- String (for hanging your creations!)

1. Planar Point Groups

This activity involves simple symmetry, measuring angles, primes, and factorization. The traditional 6-fold snowflake is just one of infinitely many possible symmetry groups that have some number of mirror lines going through a point. A common variation is the simpler 4-fold snowflake, but the simplest possible group has just one fold, making one mirror line. Before working with spheres, it's good to be familiar with snowflakes and their variations, and be confident in folding a snowflake that has any number of mirror lines. The key idea is that one triangle of paper, or one pie-slice of the plane, is the fundamental region of a symmetry group. The thing you get when you fold a symmetric thing to overlap the symmetric parts, in this case a folded-up paper triangle, is known as an *orbifold* [1].

There are some interesting quirks found when folding *n*-fold snowflakes! First, one might notice that to make a 4-fold snowflake, you do 3 folds: in half, in half again, and in half again. It's quite literally a Three Fold Snowflake. Yet when the folded sheet is unfolded, it's clear to see that it has eight pie-slice sections, divided by one fold each, so maybe it should be an Eight Fold Snowflake. This can be a point of confusion with those who are not familiar with symmetry nomenclature. The "4" comes from the four theoretically infinite mirror lines, that continue through the center of the snowflake. The third fold in the 4-fold snowflake puts a fold line through four layers of paper, each containing one half of an infinite mirror line, for a total of two more, or four total, lines. And folding it in half just once more would result in 8-fold symmetry!

But for numbers that aren't powers of two, one can't just fold in half and in half again. The 6-fold snowflake is sometimes taught as folding a paper in half, then in thirds, then in half again, but can also be taught as folding in half, then half again, and then in thirds. You'll notice the steps are commutative-- two times three equals six just as much as three times two does. And to make a 12-fold snowflake, one could start with the orbifold of a six-fold snowflake and fold it in half, or the orbifold of a 4-fold snowflake and fold it into thirds.

Figure 2: *A paper orbifold of 4-way rotational symmetry, taped and ready to cut, then ready to unravel.*

As you can see, factorization starts to play an important role when thinking about how to fold n-fold snowflakes! Dividing half a plane into six equal pieces is easier when broken down into two smaller problems, but what about prime numbers? To create a 5-fold snowflake, as in Figure 1, one must fold the plane in half once, and then into five equal sections. This can be difficult to do by hand! For higher primes, a protractor is recommended, and making prime snowflakes is good motivation if you're learning how to divide circles and measure angles!

Practical tip: fold your paper back-and-forth like a paper fan or accordion, rather than folding over all the layers at once, if you have many layers, to avoid the asymmetry created by the thickness of the paper.

Now, what about rotational symmetry? Well, the orbifolds of the rotational point groups are cones, with the rotation point at the tip. The angle of the cone depends on how many times the paper must rotate around to make it. To make an actual cone orbifold out of real paper, it is necessary to make a cut from the edge of the paper to the point of rotation, though this cut can be any shape, which is good to keep in mind if you have a plan for a final design and want to avoid taping things back together. For n rotations, curl the paper around n times, remembering that a flat sheet of paper already circles around once, not zero times. Cut the paper while in a cone as in Figure 2 (taping the end in place while you cut helps) and then unroll! Unlike folding, there's no combinatorial trick, though students may enjoy trying to figure out how to predict the angle of the cone.

2. Frieze and Wallpaper Patterns

A string of paper dolls is made by folding a strip into a zig-zag pattern, then cutting out a half-person, which is the fundamental region of a repeating string of mirror-symmetric people. Folding and cutting frieze patterns has been written about elsewhere [2] so I won't spend much time on it here, but it is worth saying that all seven frieze patterns can be folded and cut, if you're willing to use tricks like cutting a slit through the paper to get points of rotation. Figuring them all out makes a good activity for advanced students or a math club, though knowledge of the patterns and what makes a symmetry group is a prerequisite.

The wallpaper groups can also be made, though the ones that have rotation points not on mirrors are extremely tricky and require dextrous hands as well as a healthy belief that the theoretical implementation is more important than the practical result (which will probably not be very pretty). If you've got some dedicated students who care about having examples of every symmetry group, definitely go for it!

Figure 3: *6-fold sphere and its flat analog, with symmetry lines marked and finished product.*

3. Sphereflakes!

How do you fold a sphere in half? Well, if you've got uninflated beach balls, they might already come perfectly folded in half, with one half inverted onto the other. The fold, which goes around an equator of the ball, is one mirror line, making this half-beach-ball an orbifold! You could cut through both layers, and then inflate the result by inserting a fresh uncut beach ball of a contrasting or translucent color and inflating it inside. I usually cut off the inflator tube of the outer beach ball, then roll up the fresh one and put it through the hole. The result will be a spherical pattern with mirror symmetry!

As with a snowflake, you could fold your starting hemisphere in half, or thirds, or *n,* to get examples of the symmetry group *n, which is like a sphere with symmetry lines along some number of evenly-spaced longitude lines. Figure 3 shows a 6-fold example. Notice that all these orbifolds will be symmetric the long way, which means you can fold them in half the long way to get another orbifold that also has a symmetry line along the equator of the sphere. An orbifold with points of rotation along the equator can be made using the same cone method used above. Depending on how many longitude folds you have, your semi-spherical section of cone may be very long, with only a bit of cone-ness.

The important part of any orbifold is that all layers are equivalent. As long as all your layers overlap nicely, you're probably doing it right! For example, if you took a sphere that was folded along some latitude lines and the long way across the equator, and then tried to fold it in half again along a latitude line, the wider folded section of equator would be mismatched with the point of the north and south poles, and that would not be a legitimate orbifold.

The polyhedral groups are where things get really interesting! Figure 4 shows two examples. For these, I recommend first marking the mirror lines on an inflated ball, using a whiteboard marker, then deflating it and using the lines as a guide to folding. Figuring out how to draw accurate lines is itself a fun puzzle. Using translucent beach balls is helpful, especially when doing it for the first time, because you can match the mirror lines through the layers. Amazingly, with practice, even the icosahedral symmetry group can be folded out of an ordinary beach ball, and with one cut unfolded back into a star-covered dodecahedronish shape! Folding this orbifold from a sphere really gives an appreciation for how beautiful and unobvious icosahedral symmetry is. The unintuitive fact that folding sections of spheres this way leads to complete overlap is a very pleasant thing to experience with one's own hands. These images are stills from helpful videos you can find free online [3, 4]. Pure rotational groups have, to my knowledge, never been attempted with spheres, but if you do attempt them, be sure to let me know!

Figure 4: *Left: Octahedral symmetry, partially unfolded after cutting to show the orbifold of tetrahedral symmetry, a subset of octahedral symmetry. At this point, one could also make a slit to the center and curl it into a cone to get pyritohedral symmetry. Right: Folding a sphere to match icosahedral mirror lines is a daunting task, yet surprisingly possible!*

References

[1] John Conway, Heidi Burgiel, and Chaim Goodman Strauss, The Symmetries of Things, A.K. Peters, 2008.

[2] John Conway, Peter Doyle, Jane Gilman, and Bill Thurston, *Geometry and the Imagination,* online notes, http://geom.math.uiuc.edu/docs/education/institute91/handouts/handouts.html

[3] Vi Hart, Snowflakes, Starflakes, and Swirlflakes video, http://www.youtube.com/watch?v=8EmhGOQ-DNQ

[4] Vi Hart, Sphereflakes video, http://www.youtube.com/watch?v=toKu2-qzJeM