Hearing the Drum of the Rhythm

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Abstract

The title of the workshop is a twist on a question posed by mathematician Mark Kac in a 1966 paper, "Can One Hear the Shape of a Drum?" In the "Hearing the Drum of the Rhythm" workshop, the presenters will use music rhythms to model and think about the low end of the frequency spectrum for a circular membrane. This interactive, experiential workshop seeks to demonstrate the potential of the use of music to build intuitions about deep mathematical questions and of the value of music in providing "aural representations" of these ideas.

Introduction

In 1966 mathematician Mark Kac posed the question "Can one hear the shape of a drum?" in a paper by the same title [1]. Essentially, Kac asks in an intuitive manner if we can infer the shape of a vibrating membrane (such as a drumhead) given knowledge of all the frequencies at which the membrane vibrates. In mathematical terms, the natural frequencies of an object can be determined by the eigenvalues of the Laplace operator. Using these eigenvalues to determine the shape of a vibrating membrane is a rich area of research, with applications to fields as varied as number theory, statistics, geophysics, medicine, and thermodynamics, amongst others. Investigations into the properties of vibrating membranes, including those such as "fractal drums" and "hyperbolic drums", have led to interesting insights [2] [3].

Prior to the question posed by Kac, it had been determined mathematically that one can "hear" a drum's area. In 1991, mathematicians Gordon, Webb and Wolpert identified two "drums" that have equal areas and perimeters but different geometrical shapes [4]. They proved that the drums, each a multisided polygon, display identical spectra. Since this discovery, Gordon et al and others have provided further examples of isospectral (sound-alike) drums with different shapes [5] [6].

Unsurprisingly, the shape of a drum question resonates (almost no pun intended) with musicians – particularly, percussionists, of course - and others who work with sound production and analysis. The various points of entry to the question and various twists to the question suggest many possibilities for interesting collaborations,[1] as well as providing multiple opportunities for engaging learners of STEM – at different levels – around a deep and very active research area. Indeed, the richness and embedded complexity of the question and its variations hold the potential for sustained inquiry and investigation, along pathways that can lead to STEM engagement at the highest levels, including active research. Furthermore, given the near ubiquity of drums throughout the world, anchoring investigations around "the drum" opens up many avenues for making authentic cultural connections, across levels of conceptual depth and complexity.

In the "Hearing the Drum of the Rhythm" workshop, the presenters will use music rhythm - in particular, rhythms comprised of combinations of pulses - to model and think about the low end of the

[1] The present collaboration of the authors of this paper is a case in point. We anticipate a long lasting collaboration involving interdisciplinary education, research and performance.

Laplace eigenvalue spectrum for circular membranes. The purpose of this interactive, experiential workshop is to demonstrate the potential of the use of music to build intuitions about deep mathematical questions and of the value of music in providing "aural representations" of these ideas.

The sequence of activities below is designed to facilitate participants making connections between music pulses and music tones and using this understanding to determine a drum's diameter. The presenters will model a process for providing STEM (science, technology, engineering and mathematics) learners, of varying levels of preparation, access to an active STEM research area. At a more fundamental level, the presenters are reinforcing a paradigm lying at the heart of the Bridges conference, a STEAM (science, technology, engineering, art, mathematics) paradigm.

The activities are described in enough detail to allow for easy replication by others. Sound files and slides referenced in this paper are accessible at http://math.mit.edu/~tblackman/bridges2013.

Activity One – Hearing the shape of a drum

In laying a foundation for the investigation, there is value in making a connection between real drums and the mathematical, idealized versions of these drums referred to in Kac's question. Strike a variety of drums hidden from view. Discuss the question, "What can you say about the drums that you heard?" In addition to speculations about the sizes and shapes of the various drums, participants might make mention of the tension of the drum skins, about the placement of the strikes, and so on. In order to make the shape of the drum(head) salient, these various other features are not considered, which also makes the mathematics of hearing the shape of a drum more manageable.

One could say that hearing the shape of a drum involves having abstract ears able to hear an infinite spectrum of frequencies. Hearing shapes uniquely would require no two different shapes having the same frequency spectrum, since if two differently shaped drums shared the same frequency spectrum, you wouldn't be able to know for certain which was which.

Show a slide (Figure 1) of isospectral drums found by Gordon, Webb and Wolpert in 1991. Mention that the two sound-alike drums identified by Gordon, Webb and Wolpert have different shapes, but they have the same area and perimeter. Note that simply finding shapes with the same area and perimeter does not guarantee isospectrality.

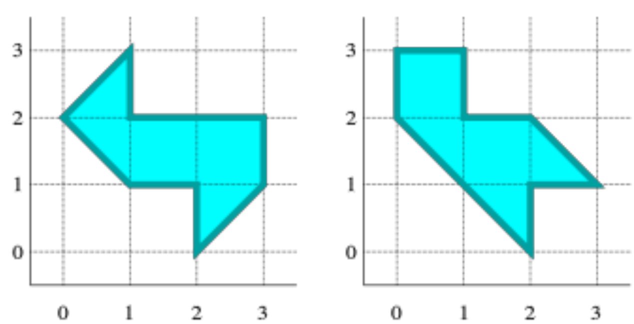

Figure 1: *Two isospectral drums that have different shapes*

"Hearing the drum of the rhythm" involves hearing the area of circular drum membranes. Participants will be using rhythms formed by combinations of pulses to model information at the low end of the frequency spectrum for these drums and to think about the relationship between the fundamental frequency and the size of a circular membrane. These concrete experiences help to develop a feel for some of the more abstract ideas embedded in Kac's question and its variations.

Activity Two – Hearing Frequency

In this activity, the group is encouraged to consider different ways of thinking about frequency and sound.

Begin with the question, "What comes to mind when you think about frequency?" Record participants' responses – on whiteboard, chart paper, or electronically. Use their responses to settle upon working definitions for frequency and cycle.

For example, some provisional definitions are: *Frequency* is the number of occurrences of a repeating event per unit of time. *Cycle* refers to one occurrence of a repeating event.

Frequency, cycle and period are some of the properties involved in talking about sound. For example, when we hear the note A above middle C on the piano, we are experiencing a sound event that occurs at 440 cycles per second.

Play Sound File #1 (A 440 Hz) and ask, "What cycle is being repeated 440 times per second?" Play Sound Files # 2 - #5 (220 Hz, 880 Hz, 110 Hz, 55 Hz). Since it isn't possible to count cycles that occur at this rate, it might be helpful to work at the level of sound that we perceive rhythmically. We can use beats per second as a way of modeling cycles per second. We can use drum to play with this idea. In doing so, we can think in terms of each beat marking the beginning of a cycle. So, for example, playing one beat per second would represent one cycle per second.

Demonstrate and have volunteers demonstrate playing the drum at different rates (1, 2, 3, …beats/second). Below is a chart of the note A, played at different octaves.

Note	Frequency
A1	55 Hz (cycles per second)
A2	110 Hz
A3	220 Hz
A4	440 Hz
A5	880 Hz
A6	1760 Hz

Ask the group what pattern(s) they notice. (Going up or down an octave doubles or halves the frequency.) Typically, we are unable to hear frequencies below 20 Hz, one reason why it can be helpful to model lower frequencies with rhythmic sound. So, for our purposes, we can consider playing 7 beats per second on the drum as a close representation of a lower octave of the note A.

Ask what it would sound like if someone were able to play the drum 440 beats per second. Play Sound File #6 (recording of drum beat starting slow and gradually speeding up to 440 beats per second). Make connections between music intervals (combinations of tones) and pulses played in combination. For example, there is a correspondence between two tones in a 3:2 frequency relationship (a so-called "perfect fifth") and pulses played in a 3:2 relationship. Have the group experience hearing/playing different combinations of pulses. This will help with understanding relationships between the natural frequencies of a string.

Activity Three – Hearing the String of a Rhythm

In this activity: (1) participants begin to explore natural frequency as it occurs in the one-dimensional case of strings, (2) use rhythm (a combination of pulse beats) played on percussion instruments to model a fundamental frequency and first few overtones, and (3) use this information to determine (the length of) the string of the rhythm.

The one-dimensional analogue to a vibrating membrane with a stationary boundary is a vibrating string with fixed ends. Similar vibrational phenomena occur in both the drum and the string. When objects are struck, plucked or given some other impulse they vibrate in ways particular to the object. These states of vibration are known as the natural frequencies of the object.

Below (Figure 2) is a chart that shows the ratios of the beginning natural frequencies of a string. The sequence of natural frequencies is referred to as the fundamental and its overtones.

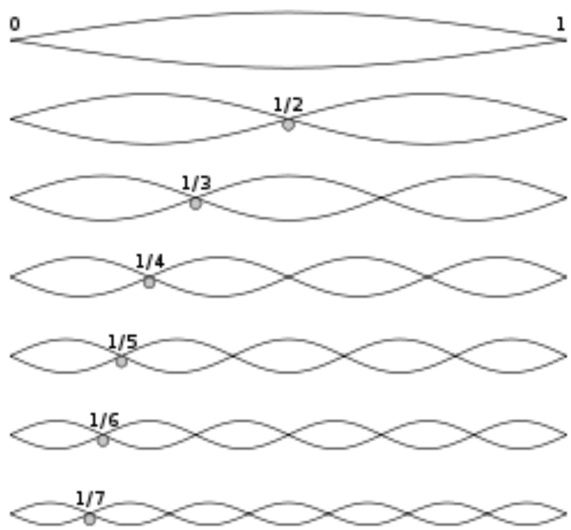

Figure 2: *Natural frequencies of a string*

Ask the group what they notice about relationships between the different frequencies. Responses might include observations about the shapes of the vibrations, and, perhaps, references to sine waves. As well, participants might notice the pattern of the higher and higher number of divisions of the string.

Acknowledge the various responses and call particular attention to the pattern of overtones arising from divisions of the string into 2, 3, 4, … parts. Explain that the frequencies of an actual string (such as a guitar string, for example) depend upon the length of the string, the tension applied to it and its linear density. However, for all strings, the fundamental and overtones are in the same ratio relationships. This series of natural frequencies is also referred to as the normal modes of a vibrating string.

Have the group experiment with using percussion instruments to play combinations of pulses in some of the ratio relationships of the vibrating string's normal modes. Play Sound File #7, representing the fundamental and overtones shown in the figure above.

Refer to a slide of the wave equation for vibrating strings, where if $u(x,t)$ is the displacement of the point x on a string at time t,

$$\frac{\partial^2 u}{\partial t^2} = k\frac{\partial^2 u}{\partial x^2}$$

$u = 0$ on the string's endpoints

Explain that using a differential equation solution approach known as separation of variables, this becomes an eigenvalue problem. The infinite set of eigenvalue solutions is referred to as the eigenvalue spectrum; the eigenvalues are the squares of the string's natural frequencies. Remind participants that the nature of the workshop is more about developing intuitions about some of the fundamental ideas versus an in-depth exploration of the formal mathematics. Nonetheless, note the remarkability of being able to represent beautiful aural and visual patterns through the language of mathematics, which, hopefully, provides motivation for unpacking the meaning and significance of these symbolic representations.

The formula for finding the natural frequencies of a vibrating string is:

$$frequency = \frac{an}{2L}$$

(where a is the velocity of sound waves traveling through the string). Work through the following example of using a fundamental frequency to determine the length of a "guitar" string.

Finding the length of a guitar string

The fundamental frequency of the G-string on a guitar is 196 Hz. Given *a* =255 m/s in the formula, find *L*, the length of the vibrating portion of the string.

Finding the string of the rhythm

Play Sound File #7, a recording of a combination of pulses. The recording includes pulses that represent the fundamental and first few overtones for a string. Ask, "What is the length of the string that would have this fundamental frequency?" Let participants work in teams to answer this question, which involves listening for the slowest pulse and determining its frequency in order to solve the problem.

Activity Four - Hearing the drum of the rhythm

In this culminating activity, participants consider natural frequency as it occurs in the two-dimensional case of circular membranes, use rhythm (a combination of pulse beats) played on percussion instruments to model the fundamental frequency along with some of the other normal modes, and use this information to determine (the diameter of) the drum of the rhythm.

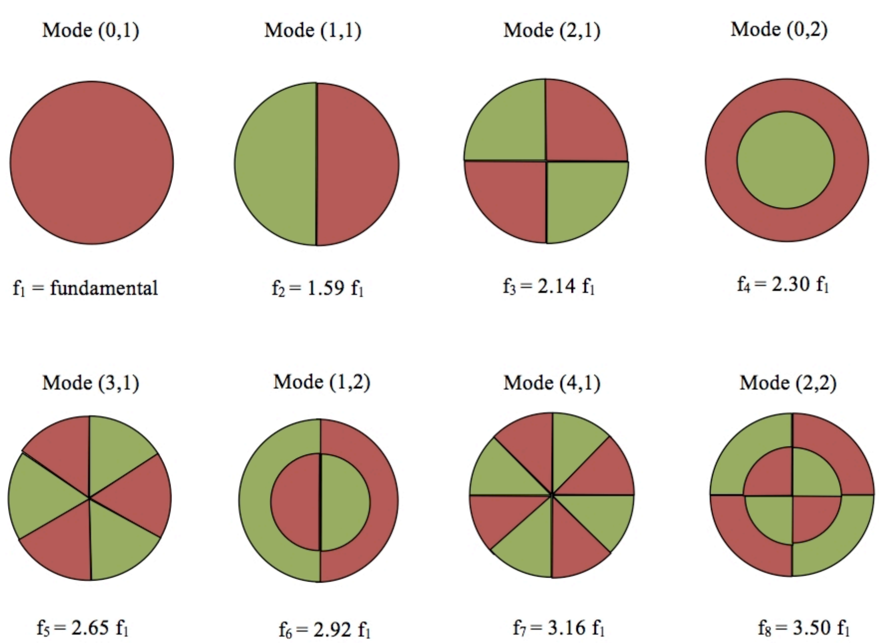

Figure 3: *Normal modes of a circular membrane (from http://homepages.ius.edu/kforinas/S/Percussion.html)*

Show a slide of Figure 3, a chart of the first 8 normal modes of a vibrating circular membrane.

Explain that these are the first eight modes of a vibrating circular membrane that is fixed at the boundary. Call the group's attention to the frequency relationships of the various modes and ask how these relationships compare with those of a vibrating string. (More than likely, participants will note that unlike the fundamental and overtones of a string, the modes of a circular membrane are in non-integer ratio relationships. Ask participants how they think this might affect the sound of a drum.)

Ask for volunteers to make attempts at playing percussion instruments that approximate the ratio relationships of different combinations of modes. Explain the mode numbers that are given above each image, represented in the form (d,c) where d is the number of nodal diameters and c is the number of nodal circles. Note that the different shades indicate that the shaded areas are moving in opposite directions. Play animations of some of the modes, for example those found at http://www.acs.psu.edu/drussell/demos/membranecircle/circle.html.

Show a slide of the two-dimensional wave equation, where if $u(x,y,t)$ is the displacement of a point (x,y) on a membrane D at time t,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \text{ in } D$$
$$u = 0 \text{ on the boundary of } D$$

Mention that the solution to this equation for circular membranes involves a class of functions, Bessel functions, which have a variety of applications in mathematics, physic and engineering.

Show a slide of the formula for finding the fundamental mode of a circular membrane.

$$f_1 \approx \frac{2.405}{2\pi a} v \approx \frac{0.766}{d} v$$

(where a is the radius of the membrane and v is the velocity at which sound travels through the membrane). Provide the following example of using a fundamental frequency to determine the diameter of a "djembe" drum.

Finding the diameter of a drum

The fundamental frequency of a djembe drum was found to be 350 Hz (http://www.drums.org/djembefaq/v20a.htm). Given v =155 m/s in the formula, find D, the diameter of the djembe. (D is approximately .34 m.)

Figure 4: *A djembe drum from Mali*

Finding the drum of the rhythm

Play Sound File 8, a recording of a combination of pulses. State that the recording includes pulses that represent the fundamental and first few modes for a djembe drum. Ask, “What is the diameter of the drumhead that would have this fundamental frequency?” Let participants work in teams to answer this question, which involves listening for the slowest pulse and determining its frequency in order to get the necessary information to solve the problem.

Conclusion and Implications

The aim of this workshop is to provide an engaging entry, suitable for learners at different levels of preparation, into a rich area of mathematics and science research. As an introduction, the expectations are modest with respect to how much and how deeply the content of spectral geometry can be covered. Nonetheless, we do seek to model practices that encourage the type of creative inquiry that leads to depth of thought and inquisitiveness about the ideas being explored. Our experience tells us that asking questions in novel ways creates opportunities for new learning, wherever one is placed upon a spectrum of understanding. “Can one hear the shape of drum?” was a novel way of asking a (set of) question(s) about sound (as eigenvalues to be heard by those who can hear infinity) and shape (in Euclidean space as well as spaces that can only be accessed by the most adept shape-shifters). In hearing the drum of the rhythm we seek to raise questions about rhythm, tone, frequency, and drums, including those that have never been heard before, except in the mind’s ear. Our hope is that participants will arrive at a place where we, the authors, find ourselves – with more questions than we started, not the type that take us around in circles (unless they are circular membranes!) but those that point in the directions of exciting paths of discovery.

Follow-up experiences we envision to this workshop include devoting time to unpacking the wave equation for a string and for a membrane. This would entail demystifying the symbolic representations and taking time to savor moments of mathematical beauty. We would find ways to play with the change of change and nudge conceptual movement in the direction of how the divergence of the gradient can tell a membrane where to get excited. It will be important to provide experiences that lead towards insight into the Bessel function, particularly given the role this function plays in the mathematics of circular membranes. Factorials and the gamma function would be a part of this journey. We look forward to meeting the challenges and opportunities of finding and/or creating aural/music experiences that help learners build foundational intuitions and that facilitate and stretch understanding.

We realize that in this workshop, though we speak in terms of hearing (the size of) a drum based upon rhythm, in fact the activities lead towards using a single pulse to determine the fundamental frequency; this relatively simple relationship gives us all the information we need to answer the size question for a circular membrane. Investigating “real” drums can make the task of hearing the drum more real. Different drums – such as timpani, tabla, djembe, atsimevu, atumpan…- sound different, in large part, because of the manner in which they are constructed, which can alter what happens with the frequency relationships of the normal modes. For example, the manner in which the drum membrane is affected by the black patch affixed to the center causes the normal modes of the tabla to be in ratio relationships similar to those of a vibrating string [5]. Different drums lend themselves to different playing techniques, which heighten or dampen the contributions of the various modes to the sound that we hear. These variations lend themselves to investigations that would give more significance to using combinations of pulses in the appropriate ratio relationships to model what is going on. Included among the variety of “real” drums are rectangular shaped frame drums, such as the tamalin drum of the Ga people of Ghana, or box drums such as those used to play one of the forms of Rumba in Cuba. Using rhythm to model the spectral qualities of these different drums present further opportunities to connect sound and shape in novel ways and to perhaps provide more evidence to support the idea that it takes a drum to know a drum.

Many interesting inverse spectral problems, arising from “hearing the shape of a drum”, remain ripe for investigation. Fascinating questions exist about what information the spectrum contains about the dynamics of a system [6]. Questions remain about the size of the sets of isospectral objects. Many interesting implications emerge from using music intuitions and practices to inform the mathematics and vice versa. For example, a question such as, “Can one hear a family[2] of drums?” is compelling[3].

A core motivation for our endeavor is an aim to get more young people, particularly those from underrepresented populations, securely on pathways to STEM professions. This entails figuring out more effective ways to engage learners around deep content, beyond attempts to simply make this content fun and friendly. More significantly, it is a matter of STEM practitioners and educators being more thoughtful, creative and resourceful in helping potential mathematicians and scientists of tomorrow find multiple entry points and pathways to excellence and high achievement. Hearing the beats of different drummers might help us walk and dance with the right moves.

References

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[8] Peter Buser, John Conway, Peter Doyle and Klaus-Dieter Semmier. Some planar isospectral domains, *International Mathematics Research Notices*, 1994.

[2] Typically, in African and African Diasporic traditions, an individual drum is a member of a larger family of drums, with their own distinctive “voices” and roles.

[3] Related to this line of questioning are Chapman’s results that show that “one cannot hear the shape of a two-piece band” [6] [7] and the work of Buser et al who identify 17 isospectral families in their paper [8].