Mat Weaving: Towards the Möbius Band

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Abstract

Traditional mat weaving, as practiced in several regions of Southeast Asia, presents the potential for mathematical exploration that can lead to the creation of plaited forms with interesting mathematical properties. This paper and its accompanying workshop introduce some of the possibilities that the writers have discovered and related mathematical properties and constraints.

Mat Weaving Using Paper Strips

This work was originally inspired by plaited reed mats produced in Southeast Asia ([1], [2]). Basic mats are made of long, narrow strips of locally available reed or leaf, whose interlacement is oriented obliquely (i.e. diagonally, or at an angle) to the exterior edges of the mat, although the local interlacement of elements (to each other) is generally perpendicular. This is commonly referred to as “mat weaving”, although this oblique technique is also often referred to as “plaiting” because it uses only one set of elements ([3], p.63; [4], p.93), to distinguish it from regular perpendicular “weaving” that uses two distinct sets of elements oriented parallel to the edges ([3], p.74; [4], p.97). Making these mats often begins at one of the corners ([4], p.165), where the elements in the set are first entered one by one until all the elements are present (see below).

More elaborate plaited mats can take advantage of two kinds of decorative effects: one based on patterns in contrasting colours; the other based on patterns of inwoven (not cut) holes in the surface creating an overall lacy effect, sometimes known as “open work” [1] or “open-cut work” [2]. This paper will focus on questions concerning the inherent mathematics of planar and related 3-dimensional *plaited* artefacts made in oblique 1/1 interlace technique with inwoven slits and holes. Colour will be used to identify each individual strand, so that its circuitous path around the form can be followed easily, and the continuity of each strand in completing its circuit can be respected.

A few definitions:
- **Strip:** a long, narrow piece of paper that functions as a single strand/element in the weave.
- **Band:** a closed circuit composed of interwoven strips; e.g. a Möbius band.

- **Arm**: locally planar, linear part of the surface of an artefact with holes or interior edges, branching away from a corner.
- **Re-entry:** a location where a strip is folded back into the weave at the edge of the plaited surface ([5], p.645).
- **Selvage:** a complete sequence of adjacent re-entries, between two changes of direction ("corners"; see Figure 3).
- **Edge:** a complete sequence of adjacent selvages, *across* changes of direction, that form a closed loop; e.g. a Möbius band would have a single edge, but could have multiple selvages.
- **Möbius band:** mathematical structure resembling a paper strip that is twisted once and closed to form a non-orientable surface. This surface has only one side and a boundary that is a simple closed curve (topologically equivalent to a circle, see Figure 1).
- **Width:** number of strips that make up a band or arm. In this paper, this will generally be 6.
- **Mat:** a planar artefact composed of interwoven strips, with an obvious outside edge.
- **Annulus:** an artefact composed of interwoven strips, bounded by two primary edges, which may be planar (e.g. like a frame) or 3-dimensional (e.g. like a two-cornered "D-shaped" form).

Figure 1: *A Möbius band*

- **Plaiting (oblique 1/1 interlace):** an interlacing of *one* set of elements oriented obliquely – in this paper at 45° – to the edges and characterised by re-entry at 90° ([4], p.93; [5], p.645).
- **Weaving (perpendicular 1/1 interlace):** an interlacing of *two* distinct sets of elements oriented perpendicularly to the edges and characterised by re-entry at 180° (Figure 2).
- **Slit:** perforation in the surface of an artefact, bounded by two selvages of different arms, whose width is 0.
- **Hole:** perforation in the surface of a planar or 3-dimensional artefact, characterised by more than 2 selvages; multiple holes create a lacy effect with multiple secondary interior edges.
- **Circuit:** the complete, closed, continuous path of an individual element/strip through the interlace.

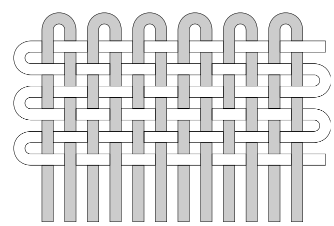

Figure 2: *Weaving (perpendicular 1/1 interlace)*

Basic Technique

Traditional mat plaiting technique ([1], [2]) has the following properties:

- It uses a single set of elements ([3], pp.60, 62; [4], p.93).
- These elements run diagonally to the edges and re-enter the mat at 90° by means of a 45° fold ([4], p.165; [5], p.645; [6], p.218).
- The completed mat has selvages on all edges ([3], p.62; [6], p.218)
- The most basic form uses an oblique 1/1 interlace, commonly called *plain weave,* in which the individual strips *always* alternate between going over, then under, successive crossing strips.

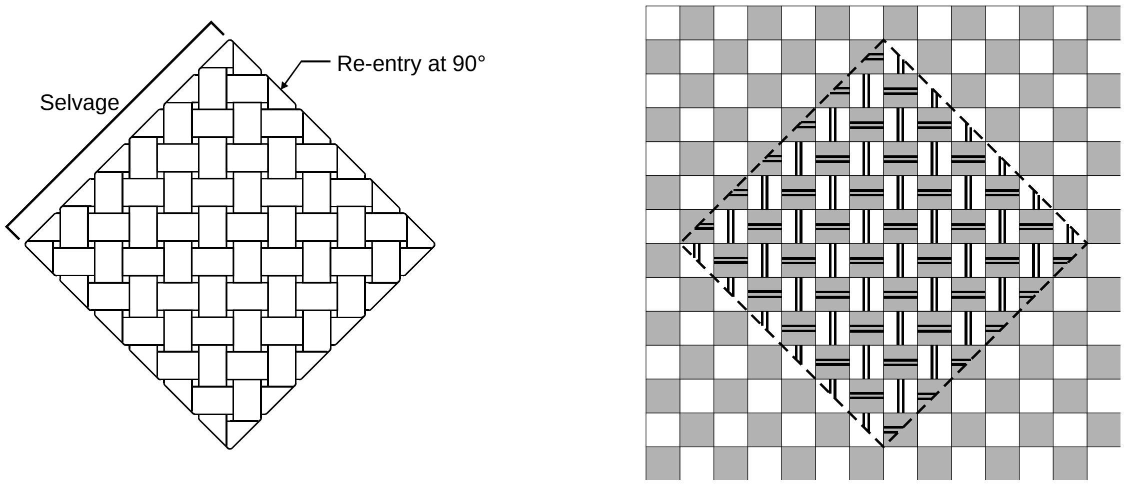

Figure 3: *A simple square mat made from 6 strips — the portion of a checkerboard that it fills*

Figure 3 shows a small, simple mat made from 6 individual strips. The four selvages are created by a technique that brings each strip back into the weave at the edges, by re-entry, changing it from a horizontal element to a vertical element or vice versa [5]. Note that in the sample of Figure 3 each strip travels along a closed, rectangular trajectory, which we call a circuit. The sample is made using 6 strips. In plain (1/1) weave each strip alternates between being over or under the crossing strip, *even on the edge*, so that the pattern is analogous to that of a checkerboard (Figure 3, right). Since the edges of the area occupied by a completed mat run diagonally to the grid pattern, adjacent re-entries on a given edge turn the same way (i.e. in Figure 3 the horizontal strips are visible on each re-entry on the edges travelling from the bottom left to the top right, the vertical strips on the other two edges).

The authors found it easiest to make this mat by beginning in a corner and working from the bottom. **Step 1:** fold a thin, long paper strip at its approximate midpoint to make a right corner (90°) as shown in the diagram of Figure 4a. **Step 2:** fold again, this time under, so that both ends run parallel, almost touching, but the left end is on top of the corner and the right underneath (4b). **Step 3:** the next strip is woven in so as to respect plain weave (4c), and **Step 4:** folded up at the edge as the first one was (4d).

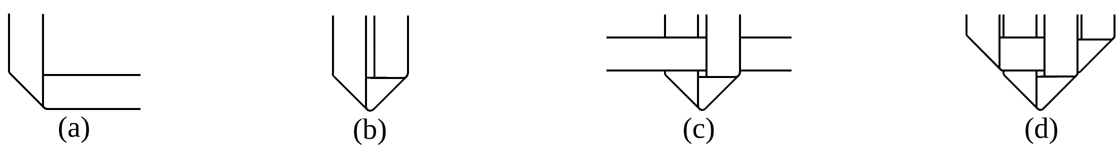

Figure 4: *Starting to weave a mat (steps 1-4)*

Steps 3 and 4 of this technique are repeated until all the strips (in the sample, 6) are woven in and folded up. The weaver has now produced half of the sample mat of Figure 3. At this point, the last strip that was introduced is re-entered into the row above at 180° (by making two successive 45° folds), so as to form a corner analogous to that formed by the first strip (see seventh row in Figure 3, left).

As the individual strips each complete a closed circuit, the ends are spliced together by overlapping them across several over-and-unders. In particular, when splicing across a re-entry, the strips are best secured by wrapping one around the other, respecting the re-entry direction. This takes some practice and will be demonstrated in detail in the workshop. The selected colours show how a single element travels along its circuit, and how adjacent strips remain adjacent on the surface: Colour A is and remains adjacent to B; B adjacent to C, etc.

Figure 5: *a simple mat with open-work*

Variations: open-work technique

As mentioned in the introduction, mat weaving can be elaborated with a lacy decorative technique sometimes called “open-cut” ([2], 2011) but better described as “open-work” (Figure 5): no cutting is involved in creating the holes; instead, the strips are re-entered locally, inside boundaries of the mat. The following variations use this technique, in each case using only 6 strips to keep the design simple.

Planar Annuli: Mats with Holes or Slits. The clearest example of open-work design involves the creation of a square mat with a square hole in the middle. Figure 6b shows this design.

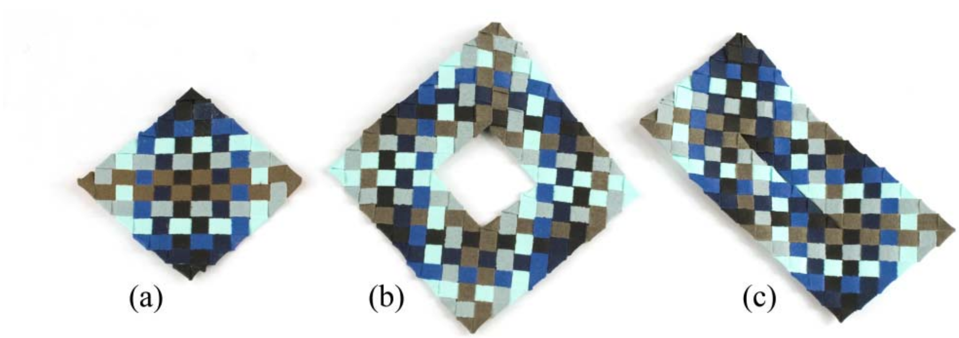

Figure 6: *Simple square mat— square with 3×3 hole — rectangle with 0×3 slit*

To create this design, the six strips are woven together as in the first 6 rows of the simple square. On row 7, things begin to change: instead of re-entering the sixth strip from the outside in, the hanging ends of the first strip are re-entered away from each other to begin creating a hole in the middle (shaded strip in Figure 7), defined by surrounding *arms.*

To make a square “frame” as in Figure 6b, the technique is repeated until there are an equal number of re-entries on each inside selvage, in this case, 3 re-entries. Note that this begins the weaving of two arms that meet at the corner formed in the beginning. When the internal re-entries have been completed on each arm, a corner is turned as in the simple square mat by re-entering on the *outer* corners of the woven area. After, in this case, 3 more internal re-entries on each arm, the ends are spliced together (by overlapping as convenient to close all the circuits, reprising the method of weaving the first 6 rows, but in reverse, to end with a corner.

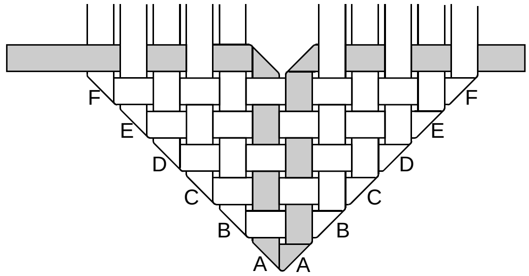

Figure 7: *Creating a hole by re-entering the ends of strip 1 in opposite directions*

In Figure 6c, this same method was used, but only one arm was given 3 internal re-entries, whilst the other was given an immediate corner, then, after the two corners, the opposite inside selvage has 3 re-entries, before the annulus is closed with a corner, forming a mat with a 3×0 slit.

Non-planar artefacts. The above examples have so far all been able to lie flat in the plane. It is possible to create artefacts that will not do so, by playing with the (relative) number of re-entries on each selvage. In a later section, we will describe the mathematical constraints that dictate this, whilst preserving the continuity of colour.

The simplest way to create a non-planar artefact (*shape*) is to vary the lengths of the various arms. The non-planar property is then derived from the fact that the technique imposes 90° angles at the corners, whereas the lengths of the arms would create an irregular quadrilateral. For simplicity, we describe the lengths by referring to the number of *internal* re-entries on each band. Figure 8 shows annuli with arms, listed counter-clockwise from the bottom (left) corner: 3, 3, 9, 3 (a) and 0, 8, 4, 8 (b). In each case, the maker can track the length of the internal selvage before turning the corner, then splice the ends. We encourage the reader to work out how to make these models.

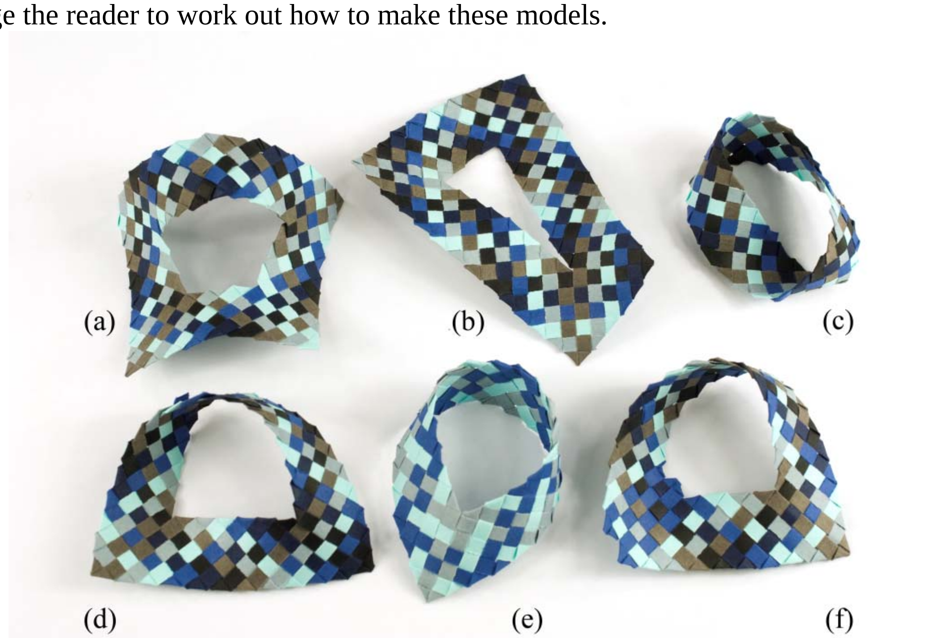

Figure 8: *Variations based on the open-work technique*

Twist into the Möbius Band. The twist inherent to the Möbius band is accomplished at the point in the process when the two arms on either side of the hole are to be joined to complete the shape. If colour is to be preserved, this can only be accomplished in specific circumstances, which will be analysed in a later section. Regardless of the presence of a twist, if colour is to be preserved across the join, the order of colours of the joining edges must both be the same, and be at the same point in the sequence, at the join. In other words, if one branch ends in ABC (coming from the left), the other *must* end in FED (coming from the right), in order for the colour sequence ABCDEF to traverse the join, thus making it invisible.

Mathematical Analysis

An individual element of a mat forms a circuit (a complete, closed, continuous path). This is emphasized in the examples through the use of distinct colours for each circuit. If a shape is begun with the strips coloured ABCDEF (see diagram in Figure 7), in its finished state, the traversing of that corner will yield the following order of colours: FEDCBA^ABCDEF (on the outside), with the corner proper (^) between the two As. In effect, turning a corner not only repeats a colour, it also reverses the order. This is a key feature of this technique and has consequences with regards to the constructability of shapes with invisibles joins, *where colours are consistent across a join*.

Colour order reversal, of course, is a reciprocal relationship in that "reversing twice" is equivalent to not reversing at all, thus, turning two corners cancels out any colour reversal. In addition, locally, both selvages of an arm show the same colour order, shifted by 3 colours. A Möbius band is therefore possible, either if there is no corner at all (Figure 8c), or if there are an even number of reversals (Figure 8d).

It is possible to circumvent the colour continuity issue and make shapes that do not have an even number of corners by using a colour sequence that has an inherent self-reversal (e.g. ABCCBA). Note that in this case a given circuit goes around the shape twice, visually producing an illusion of continuity that preserves the concept of the Möbius form with an invisible join.

Assuming the use of such an appropriately symmetrical colour sequence, the Möbius twist can be introduced into several shapes, with either equal or unequal lengths of arms, and even with jogs or hiccups, so long as at least one of the arms is long enough to make it practical for the maker to work the shape without undue stress on the strips. The shortening effect and torsion force of the twist on the arm in which it is made has to be taken into account; arms that are too short are extremely difficult to hold together and manipulate while joining the arms. For example, it is awkward to make a symmetrical 2-cornered annulus with two equal arms unless the arms have an interior edge of minimum 6 re-entries; 12 re-entries are more workable, especially with the twist. Figure 8d is a two-cornered "D-shaped" annulus of unequal arms (6, 12) with a Möbius twist. Figure 8e is a 1-cornered drop-shaped annulus with a twist that uses the colour sequence ABCCBA. Figure 8f shows an unequal, 2-cornered annulus with a jog at one end.

Because different colours are used for each circuit, the artefacts become visual examples of modular arithmetic. Modular arithmetic is a generalization of the mathematics of a 12 hour clock [7]. For example, seven hours after 9 o'clock is 4 o'clock. This is because $7 + 9 = 16$ and 16 has a remainder of 4 when divided by 12. In the clock system, 12 is referred to as the modulus, and one can perform the mathematical operations of addition and multiplication by thinking in terms of the remainder upon division by 12.

To generalise the clock arithmetic for other moduli, we use the notation $a \equiv b \; mod \; m$ (read "a is congruent to b modulo m") to mean that a and b have the same remainder upon division by m (where we assume that m is a positive integer). An equivalent definition: $a \equiv b$ mod m if and only if m divides $a - b$. Thus, $16 \equiv 4\ mod\ 6$. Similarly, $15 = 1\ mod\ 7$.

Let $\boldsymbol{Z}_m = \{0, 1, 2, \dots, m-1\}$. This set of integers consists of all possible remainders upon division by m, and is referred to as "$\boldsymbol{Z}\ mod\ m$". It is possible to visualise this set by using a clock with m hours; then the numbers wrap around after reaching the modulus. Figure 9 displays a clock for modulus equal to 6, showing how the integers up to 17 wrap around. Modular arithmetic is useful for applications such as cryptography, computer science and chemistry.

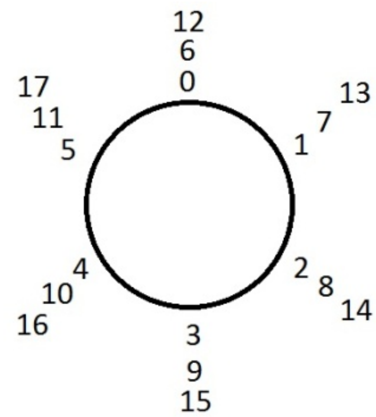

Figure 9: *Mod 6 Clock*

The colours on either side of an arm go in the same order with a phase shift of 3. This is unaffected by any twist given to the arm. An inside selvage (between inside corners) is always 6 re-entries shorter than its associated outside selvage. In this case, it is feasible to use the number of re-entries on the inside selvage because the outside selvage is always 6 re-entries longer. A selvage between an outside and an inside corner (hybrid selvage) is the same length as its associated hybrid selvage.

As mentioned earlier, turning a corner reverses the colour order. Artefacts that respect colour continuity in their circuits obey the above principles, but are affected by the number of corners, i.e. the number of reversals to their colour order. Because the colour order reverses at each corner, the colour identity at a given position is determined by alternately adding and subtracting the consecutive number of re-entries of the edges up to that position. In Figure 8a, the inside selvages are 3, 3, 9, 3 re-entries long (beginning in the bottom left corner). Thus, for a complete trajectory, this sum becomes $3 - 3 + 9 - 3 = 6 \equiv 0\ mod\ 6$. In the example on the right, the lengths are: 6, 18, 12, 18, giving a formula of $6 - 18 + 12 - 18 = 18$ which is also $0\ mod\ 6$.

With odd numbers of corners, the number of order reversals creates a conflict where the strips are joined up. The way around this is to use a palindromic sequence of colours (i.e. with internal reflection symmetry); thus, for 6 strips: ABCCBA (see Figure 8e).

Conclusion

The examples shown in this paper are merely the tip of the iceberg in terms of possible variations. Further dimensions of variation include:

- Changing the width
- Changing the distance between corners
- More corners
- Making more branchings and joins
- Plaiting together multiple plaits, creating a fractal structure of plaits of plaits of plaits…
- Introducing jogs or "hiccups" where corners are very short-lived, with a single internal re-entry, or two (see Figure 8f)

Figure 10 below shows one such new direction: the creation of a planar shape that crosses itself into a "figure eight" before joining up. This could also have been done by interlacing the strips together without interlacing the arms at the cross-over point.

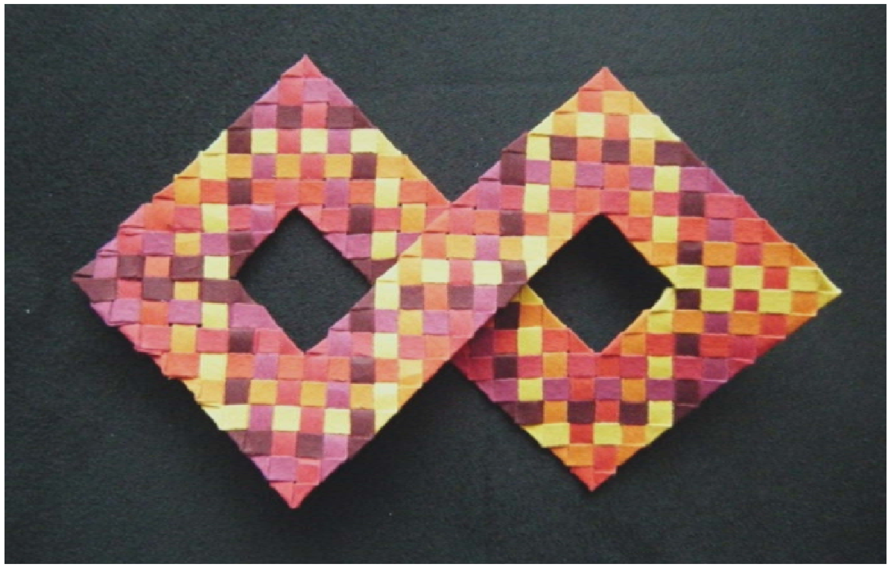

Figure 10: *"Figure eight"*

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