# Rendering 3D Tessellations with Conformal Curvilinear Perspective*

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## Abstract

Traditional perspective painting projects the world from the eye-point onto a single rectangular screen. In contrast, this article works with spherical images, created by projecting the world onto a closed sphere surrounding the eye-point, the *viewable sphere*. We focus on the use of stereographic projection of the viewable sphere to produce flat images which are conformal and include arbitrarily large fractions of the viewable sphere. We describe a practical, real-time implementation of this projection embedded in a general-purpose visualisation system. We then apply the method to provide a new way of visualising tessellations of three-dimensional euclidean (and non-euclidean) space, and give a preliminary discussion of the results.


**Motivation.** This work was inspired by the work of the American artist Dick Termes [9]. His desire to "paint the total picture" led him to paint on spherical canvases (Figure 1). The image he paints is that which an observer located at the center of the sphere would see. We call such an image a *viewable sphere*. Simultaneous with this artistic development, the last few years have seen an explosive growth in the use of such viewable spheres in photography, where they are known as *spherical panoramas*. A natural question is "What sorts of flat images can be produced from viewable spheres?" This question is the impulse behind the present research.

**Looking out from the viewable sphere.** If you could position yourself at the center of one of Termes' spherical paintings, then you would see what the artist saw when he painted the sphere. If you then insert a rectangular screen before your eye, perpendicular to the line of sight and project the sphere onto this screen, you would obtain a perspectively correct image of the original scene. However, as you move away from the center of the sphere, such projections no longer map straight lines to straight lines.

**Stereographic Projection.** The projection obtained when the eye or camera reaches the surface of the sphere itself, is a famous mathematical object known as *stereographic projection*, a conformal (angle-preserving) map of the sphere minus one point (the center of projection) onto the plane. Figure 2 illustrates the process of stereographic projection of three mutually orthogonal great circles on the sphere onto the ground plane. One half the sphere has been removed to aid the understanding. The projection of a point is

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**Figure 1** : *Two views of the spherical painting* Cubical Universe *by Dick Termes.*

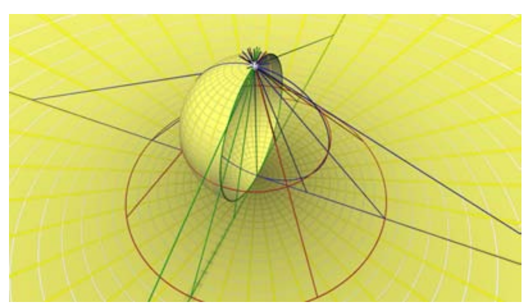

**Figure 2** : *Stereographic projection of sphere (only half shown) onto plane.*

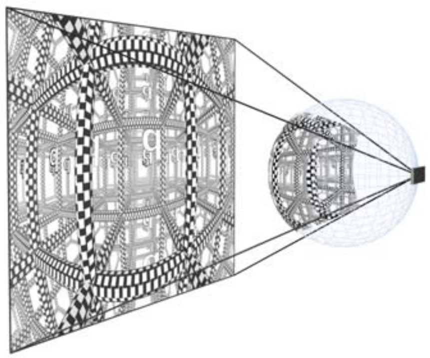

**Figure 3**: *Stereographic projection of a viewable sphere using on-axis camera.*

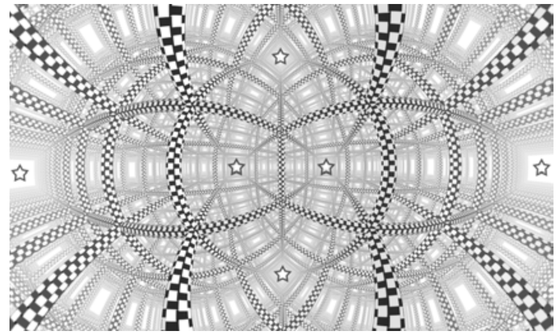

**Figure 4**: *Six-point perspective: Stars mark vanishing points in 6 cardinal directions* $\pm x, \pm y, \pm z$.

found by joining the point with the north pole of the sphere with a line. Where this line cuts the plane is the projection point. Stereographic projection has many attractive properties besides conformality, for example, it also maps circles to circles (when one extends the definition of circle in the plane to include straight lines)!

**Previous work.** [2] was a ground-breaking work that introduced the term "curvilinear perspective", and discusses practical and aesthetic issues involved in flattening spherical images. A more recent survey on the same theme is [3], which focuses on strategies for processing panoramic photographic images. Both of these sources present stereographic projection as a preferred solution. The open software project Hugin (hugin.sourceforge.org) provides an environment for processing spherical panoramas. What is new in the present treatment is the description of a practical, real-time implementation for synthetic image generation, and the focus on the resulting visualisation of 3D tessellations in euclidean and non-euclidean spaces.

**Terminology.** Termes describes his spherical paintings as "six-point perspective", since all six cardinal directions – front, back, left, right, up and down – appear as vanishing points on the viewable sphere. In the interests of brevity, we employ the same term to refer to the images obtained from the viewable sphere via stereographic projection in this way. See Figure 4. See also Figure 5, which labels the six cube faces as $(FT, BK, LF, RT, UP, DN)$.

**From viewable sphere to six-point perspective.** Assume one has a viewable sphere represented in the computer as a sphere with the projection of the surrounding scene "painted" on it (typically in the form of *texture maps*). To obtain six-point perspective, one renders this sphere traditionally, using an on-axis camera positioned on its surface, pointed at the center of the sphere, yielding a stereographic projection. See Figure 3 which shows a relatively small field of view of 60°. The six-point perspective images shown in this article have been rendered with field of views of around 120°. Rotating the sphere while leaving the camera fixed produces a family of images in six-point perspective, typically with interactive frame rates.

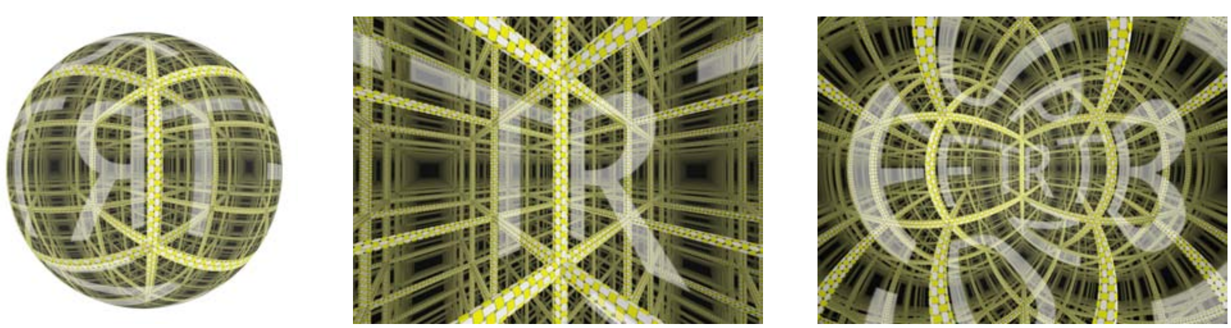

**Figure 5**: *Viewable sphere of a tessellation with standard and six-point perspective views.*

**Generating viewable spheres using synthetic imaging.** In the absence of paint or photographs, one can generate viewable spheres using standard visualization systems. Assume one has some virtual scene represented in the computer. To create the viewable sphere, one positions the camera at the origin and generates a series of ordinary perspective images of the scene that can then be "sewn together" to create a viewable sphere. This is most simply achieved by rendering the scene six times onto the six faces of a cube centered on the eye point. That is, use an on-axis camera with a field of view of 90° and square viewport. These resulting images can then be texture-mapped onto the unit sphere. See Figure 5, left.

**Tessellations of 3-space.** We use tessellations of three-dimensional space as a source of synthetic imagery. We have found that tessellations and six-point perspective are good partners. The regularity of the tiling helps the observer to understand the projection. In the other direction, the technique provides a way of seeing the "whole tessellation" which ordinary perspective rendering cannot provide. Most of the tessellations in this article come from the 10 Euclidean manifold groups (also known as platycosms) [1]. We also include some non-euclidean tessellations. The *120-cell* is a tessellation of the 3-dimensional sphere by regular pentagon dodecahedra with dihedral angles of $\frac{2\pi}{3}$; as partner we present a hyperbolic tessellation whose tile is also a regular pentagon dodecahedron, but with dihedral angles of $\frac{\pi}{2}$.

**Visualization technique.** We create a fundamental region (a single tile) for the tessellation and display it in two forms: first we thicken the edges into beams and apply a woven texture map to this geometry; secondly, as a scaled-down polyhedron. We often replace the latter with an asymmetric, texture-mapped letter such as 'd', which tends to differentiate the types of symmetries better than the fundamental domain. For more on visualization techniques see [4]; for the current state of metric-neutral visualization see [5].

**Software.** The images were created with a plugin module built into the 3D, Java-based, metric-neutral scene graph package jReality [7]. It has been optimized so that the whole process is performed on the GPU. This allows multiple frames per second with a texture resolution 6 x 1024 x 1024. For a fixed viewable sphere the frame rate is much higher. The reader can play with *SimpleManiview* [6], the program used to generate the figures in this article; it is available as a Java webstart application at the URL given in the citation.

**Discussion of figures.** As a general rule, the circles in the six-point perspective images correspond to straight lines in the original scene; a complete circle bounds the image of a plane in the original scene (see Figure 6, right). Figures 5 and 6 show views of three euclidean platycosms $c1$, $c6$ and $+a2$. Figure 7 shows the standard three views of the viewable sphere for the 120-cell. Finally, Figure 8 shows the hyperbolic tessellation by pentagon dodecahedra mentioned above. In the six-point perspective image one can see 11 of the 12 faces of the dodecahedron surrounding the camera. For further images, see the image gallery [8].

**Conclusion.** We have demonstrated a simple and effective implementation of conformal curvilinear per-

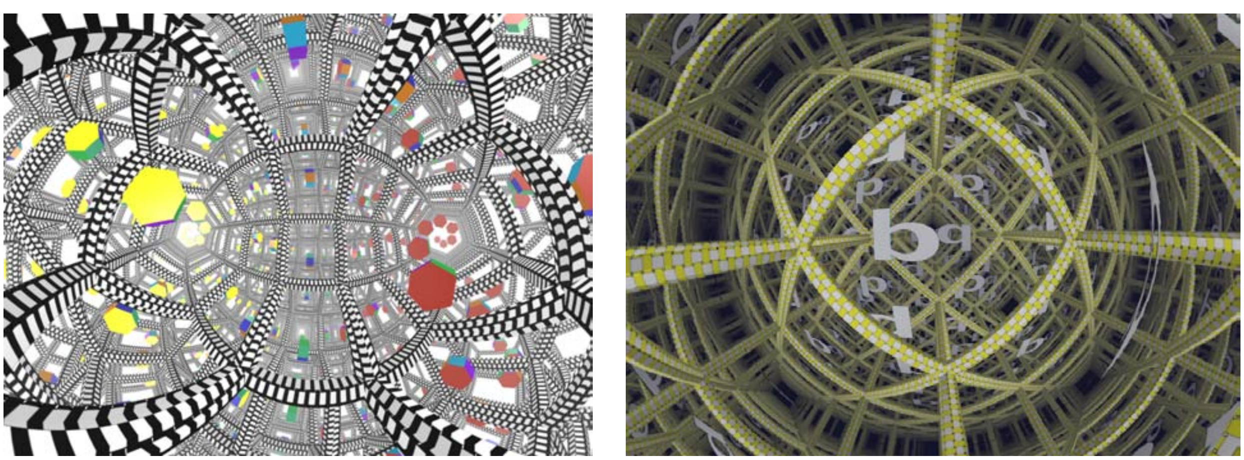

**Figure 6** : *Six-point perspective views of the euclidean platycosms* $c6$ *and* $+a2$

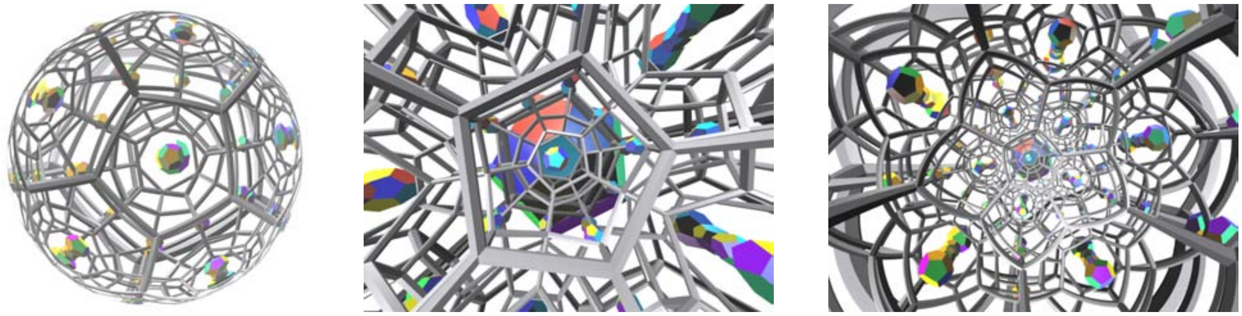

**Figure 7** : *Outside, inside, and "on" views of the viewable sphere of the 120-cell tessellation.*

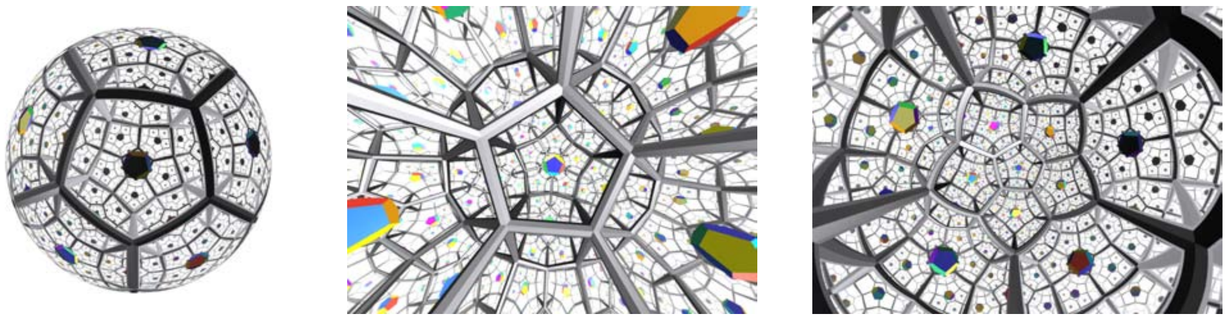

**Figure 8** : *Outside, inside, and "on" views of the viewable sphere of a hyperbolic tessellation.*

spective, which can be built into existing visualization systems. We have applied this method to visualize a variety of tessellations of euclidean and non-euclidean 3-space, resulting in images that provide new insights into patterns and aspects not accessible via traditional perspective. This experience, along with the growing popularity of panoramic photography, leads us to suggest that six-point perspective may be a perspective projection particularly suited to the current *Zeitgeist*, since it provides a global, holistic view of (almost) the whole world rather than the limited, isolated rectangle provided by standard perspective rendering.

**Acknowledgements.** The impulse for this work came from Dick Termes' spherical paintings. The tessellations were inspired by Bill Thurston (1946-2012). Thanks to the referees for valuable comments.

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