Polyhedra with Folded Regular Heptagons

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Abstract

It seems that polyhedra with regular heptagons are hard to find if the prism-based polyhedra are excluded. To be able to find attractive polyhedra that can be used for artwork it is argued to release the flatness constraint, which requires planar faces. If the regular heptagons do not need to be flat, they are allowed to be folded. Then they are orbited and holes are filled by triangles. One computer program was developed to model this and another to search for special positions. Some examples of the special case, for which the polyhedron only consists of (folded) heptagons, are shown. These were built out of Chromolux paper.

Introduction

In the field of polyhedra, regular heptagons are often disregarded. They do not seem to fit very well in 3-space. For instance none of the Johnson solids [5] contain any regular heptagons. It seems that the only way to get around this, is to use the 7-fold prism or anti-prism symmetry. Some surprising examples are given in [6] on pages 65 – 69, where prisms are used as building blocks to create polyhedra that also have prism symmetry. As an artist the author is not very interested in prism symmetries, which can be considered as promoted 2-space symmetries. The author prefers polyhedra with symmetries that are based on the tetrahedron, octahedron and icosahedron, which could be referred to as *pure* 3-space symmetries. Of these the author prefers symmetries that only consist of direct isometries, because these tend to give a twist to the model. As Figure 1 shows, within these pure symmetries polyhedra that just have regular polygons, including heptagons, do exist. However, the polyhedra still consist of heptagonal prisms glued on the squares of some Archimedean solids, which is not very interesting.

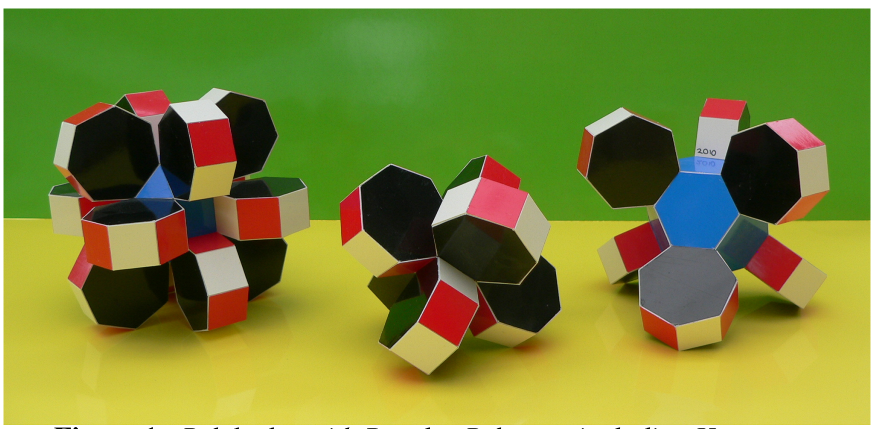

Figure 1 : *Polyhedra with Regular Polygons including Heptagons*

To get further in the quest the author recognised the need to release some constraint, as this was done in [1] and [8]. In 2008 at the Bridges conference in Leeuwarden F. Göbel gave the author the right candidate. He presented some models that he called Follyhedra: polyhedra that just consist of squares that are folded over one diagonal. He expressed his surprise that where connecting flat squares in 3-space would just lead to variations of cubes, folding the squares opened up a complete new dimension of possibilities. This is how the author started folding regular heptagons in 3-space.

Investigating Possible Polyhedra

To investigate the possibilities the author added a module to the program Orbitit that the author developed, see [7]. The program allows polyhedra to be loaded and then the user can rotate, and zoom in and out to examine them. It is also possible to print the pieces that are needed to build a model. The module that was added enables the user to investigate polyhedra with folded heptagons interactively, by folding the heptagons and varying angles and distances. This way the user can build up an understanding of the possibilities.

In the module a polyhedron with folded regular heptagons is constructed in three steps, which are merely for the sake of illustration. First the heptagon is placed in 3-space. Slide bars[1] can be used to:

- Change the distance d to the origin.
- Rotate the heptagon around one special edge E (explained below). The rotation angle is referred to as angle δ
- Rotate the heptagon around an axis through the origin and the centre of edge E. This rotation angle will be referred to as angle ϕ.
- Fold the heptagon.

In the next step symmetry can then be obtained by orbiting the heptagon. The search area was narrowed by placing the heptagon in such a way that two heptagons share an edge. This is the special edge E around which you can rotate by angle δ, and edge E is positioned in such a way that the centre of E will be on a 2-fold axis of the final symmetry.

Finally, to get a closed polyhedron the gaps between the pairs of heptagons need to be filled, e.g. by triangles. Figure 2 shows these steps from left to right including the special edge E and and its centre that is shared by the 2-fold axis.

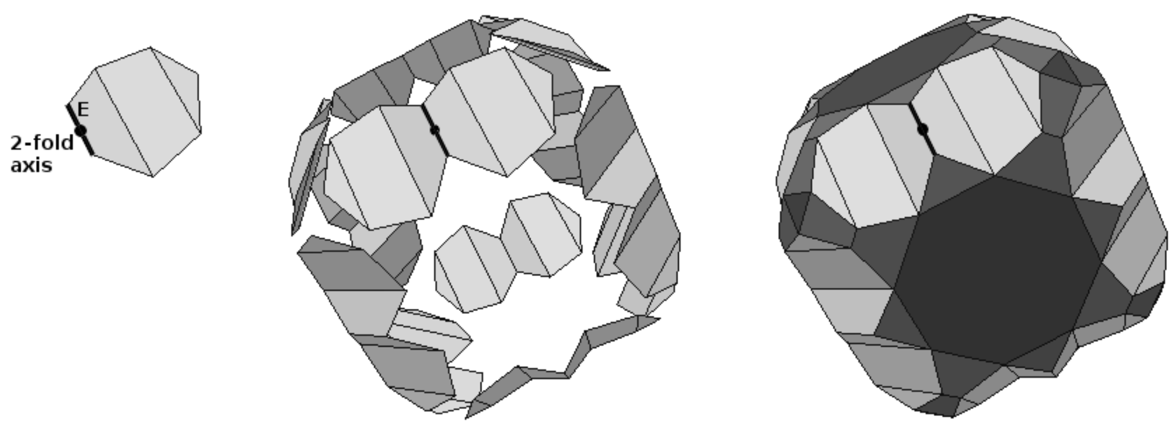

Figure 2 : *Steps to Create a Polyhedron with Regular Folded Heptagons*

After filling all gaps you can use the slide bars in the program and try to find special positions. E.g. there might be positions for which all triangles disappear, thus leading to a polyhedron that just consists of

[1]The slide bars are digital, which means that they use a certain step size. Furthermore, the module shows distances with a precision of two decimals As a consequence it is not possible to find any accurate polyhedron using the module. The next section will introduce a more accurate way of generating possible solutions.

folded regular heptagons; For some of the steps the user needs to make some more choices. The following paragraphs give some more details about this.

Folding the heptagon can be done in different ways, and which folds are possible, depends on the symmetry. If the symmetry includes opposite isometries, e.g. S_4A_4, $A_4 \times I$, $S_4 \times I$, and $A_5 \times I$ [2], then the 2-fold axis at edge E is lying in two planes of reflection (as in Figure 2). To make sure the correct symmetry is obtained the stabiliser of the heptagon needs to include the symmetry group C_2C_1, consisting of the identity and a reflection. Since the folded heptagon needs to preserve this reflection, the number of possible ways of folding is limited to the ones as shown in Figure 3. Note that, for these symmetries it is not possible to rotate the heptagon freely around the 2-fold axis through the centre of edge E, i.e. angle ϕ is constant. Note that there are two folding angles, even though the number of diagonal folds may be more than two. This has to with the C_2C_1 requirement. This set-up leads to a system with five variables: d, δ. ϕ. α, and β, where α and β refer to the diagonal folds.

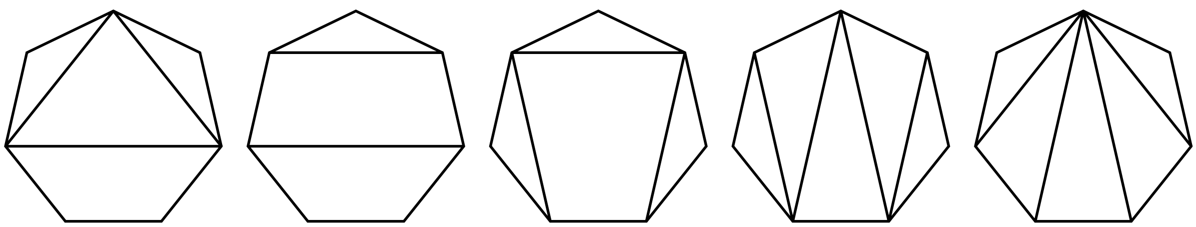

Figure 3 : *Possible Folds for Heptagons with a Vertical Reflection Axis*

In case of the symmetries A_4, S_4 and A_5 the heptagon can be folded in any possible way. This leads to quite many possibilities, which have not been examined completely yet. Instead only the folding possibilities as shown in Figure 3 have been investigated. Of these only the shell-fold and w-fold make sense, since the others have less folds, which leads to less variables. In general this set-up leads to seven requirements for edges, i.e. seven equations with up to seven variables. Since in general the equations will be independent, it would mean that there are no solutions if there are less than seven variables.

Depending on the symmetry the investigation of the angle ϕ can be reduced to a domain smaller than $[0°, 180°]$. The border values of this domain are the special cases leading to the symmetries that include opposite isometries; for the other angles the symmetry only consists of direct isometries. For the tetrahedral based symmetries it is enough to investigate the angle $\phi \in [0°, 45°]$. For the upper limit the S_4A_4 symmetry is obtained, while the lower limit leads to the $A_4 \times I$ symmetry. For all other angles the polyhedron will have the A_4 symmetry. For the octahedral based symmetry it is enough to investigate the domain $[0°, 90°]$. Both border values lead to the $S_4 \times I$ symmetry; for all other angles the polyhedron will have the S_4 symmetry. Similarly for the icosahedral based symmetry it is enough to investigate the angle $\phi \in [0°, 90°]$, while both border values lead to the $A_5 \times I$ symmetry, and the other angles the polyhedron will have the A_5 symmetry.

Filling up the gaps is done by adding triangles. This can be done in many different ways. Figure 4 shows a few examples. More possibilities exist. For symmetries that only have direct isometries the left part of the triangle fill does not need to be the same as the right part. For instance one might combine 'strip 1' and 'strip 2'. Such combinations were investigated as well.

It is difficult to find all possible ways to fill the gaps with triangles. For instance one way of filling the gaps for a certain ϕ makes perfectly sense, while it would not at all for another value of ϕ. This can be imagined by turning the pair of heptagons and the single heptagon in Figure 4 clockwise. Many of

[2]The author uses the notation for symmetry groups as defined in [2], where A_4, S_4 and A_5 consist of all the direct isometries of the tetrahedron, octahedron and icosahedron respectively, and $G \times I = G \cup G \cdot I$ and $HG = G \cup (H - G) \cdot I$, where I represents the central inversion. This means that S_4A_4 represents the complete symmetry group of the tetrahedron.

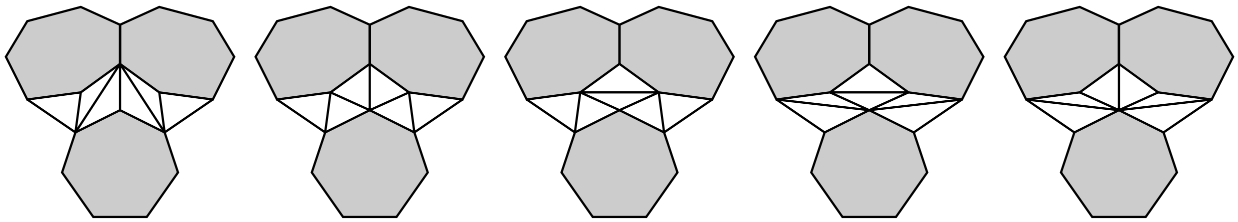

Figure 4 : *Example Triangle Fills for* $\phi = 0$

such alternative fillings have not been investigated yet. Furthermore there are complex ways of connecting heptagons that are not neighbours leading to self-intersections. These have not been considered either, though it must be noted that negative values for the distance d can give the same effect, and these *were* considered.

Figure 4 shows that several edges of variable length were added; these are the triangle edges that are not shared with a heptagon. For each of the possible triangle configurations shown, seven of these extra edges were added. Special positions can now be recognised by investigating what happens with the lengths of these seven edges, while changing the variables. E.g. if all edges in Figure 4 have a length that equals 1, then all triangles become equilateral. For the ‘strip 2’ triangle fill all triangles disappear if the length of these seven edges from left to right equal 1, 0, 1, 0, 1, 0, 1 respectively (assuming that the edge length of a heptagon equals 1). If all triangles disappear, it does not really matter which triangle fill was defined from the beginning. The same solutions are found for e.g. the ‘strip 1 loose’ fill by requiring the edges to have the lengths ‘1, 0, 1, ρ, 1, 0, 1’, where ρ is equal to the length of the shortest diagonal of a heptagon. Note that not all combinations make sense, e.g. it is impossible to keep the heptagons and require that all seven edges have length 0. The special edge lengths that the author investigated for this paper are combinations of 0 and 1.

Generating Polyhedra

The Oribitit module described in the previous section allows a user to dynamically deform the polyhedron to build up an understanding and to investigate the possibilities. However to find more accurate special positions another program was developed. The result of this program is a set of files that can be read by the Orbitit module, so that the user can have a look at the solutions. Each file represents a set-up. A *set-up* here consists of:

- A certain way of folding the heptagons.
- A certain way of filling the gaps with triangles.
- A certain requirement for the resulting edge lengths.
- A certain symmetry.

The program that generates the files uses the Newton-Raphson method to find a particular special position, meaning a position for which each edge length is equal to some specific value. This method expects the variables to have a starting value that is close to a solution. These starting positions are just guessed by generating them randomly within a certain domain. The program reads from file the current solutions for a certain set-up, tries a certain number of random starting positions and then continues with the next setup.

The set of solutions that is generated by the program has a certain floating point precision that is a parameter to the program. Even though this will lead to more precise solutions than with the module of the previous section, it does not proof whether the solution is an exact mathematical solution. This requires an algebraic proof, which is not presented in this paper.

Some set-ups lead to an infinite number of solutions. These are referred to as solutions with 'transformational' freedom. Loosely defined this is if the input parameters can be changed a little bit, so that a solution with the exact same edge lengths is found, similar to the original 'polyhedron'.

Polyhedron is written between quotes, since some of the solutions would not normally be considered as valid polyhedra, because some faces coincide. Figure 5 shows one example in the middle, where there are flaps that could be bent. Grünbaum motivates in [3] how such polyhedra can be accepted. The transformational freedom comes from folding these faces that coincide around the edge that is shared by four faces. As soon as all flaps are bent a bit by the same angle (less than 180 degrees) then the reflections are lost, but the model still has A_4 symmetry.

Other solutions with transformational freedom were found; solutions that do not have faces that coincide. For these a new valid solution can be obtained by varying the angle ϕ a little bit and by adjusting some of the *other* input parameters. So far all of these have self-intersections, which means that it is not possible to build a model that can be deformed dynamically, which would have been interesting from an artistic point of view.

Figure 5 : *Some Models with Only Heptagons:* $A_4 \times I$

Some Results

Excluding the ones with transformational freedom, thousands of models were found. So many of them that the author did not load all of them into Orbitit to have a look at them. Many of these models are hard to understand, and just look chaotic.

Some people suggested to write a program that could filter out these unattractive solutions. The problem arises then how to decide which are unattractive, or which are attractive. It is the author's opinion that this needs to be done by a person. Some polyhedra are attractive because of the intersections, which adds the right amount of complexity, others are attractive because they lack intersections and they surprise by their simplicity.

This section shows some examples from the small sub-set that only consists of heptagons. Surprisingly none of solutions that were found for the A_5 symmetry consisted of heptagons only. Though the author is hopeful to find solutions for this symmetry by using different set-ups.

Figure 5 shows some simple models that the author built. They all have the same symmetry, $A_4 \times I$, they only consist of folded heptagons and the faces do not intersect. This means that no special templates are needed and anyone, who can draw a heptagon, can build these by looking at the picture. The one in the middle of Figure 5 is a bit more interesting model from an artistic point of view since it explores the limit of the definition of a polyhedron as described before. From left to right these are examples of a shell fold, a parallel fold and a W fold.

Figure 6 shows some more examples of models with the higher order symmetries S_4, $S_4 \times I$ and $A_5 \times I$. The one in the middle, using a shell fold, is another example for which pairs of triangles coincide and form flaps. The other faces do not coincide. i.e. the 'polyhedron' does have a positive volume. The polyhedron on the right has $A_5 \times I$ symmetry and it is obtained by folding the heptagons according to the W fold method. No faces coincide for this model, even though the figure might give the impression. It reminds the author a bit of a flower. The well-known example of nature showing us the golden ratio is the sunflower, for this reason the author refers to this one as the Moonflower: not only does it reflect the golden section (because of its symmetry) it also includes similar ratios for the heptagon, see [4]. The one on the left is another example of a shell fold. It only has direct symmetries and belongs to S_4. The author prefers models that lack the reflections, since they give a twist to the model. These are often more challenging to build as well. The nice thing of this model is that it is hard to see that the model consists of folded heptagons. Besides that it is a nice example of a model with some intersections.

Figure 6 : *Some Models with Only Heptagons:* S_4*,* $S_4 \times I$ *and* $A_5 \times I$

Finally Figure 7 shows some more examples of models with intersections. The author especially likes the one on the left, which has $A_4 \times I$ symmetry. For this one it is also hard to recognise that the model consists of heptagons. Once again this is an example a folding the heptagons according to the shell fold. It is sort of an anti-star, where the arms of the star point inwards, but so much that they point through the whole model. For the model on the right, which has $S_4 \times I$ symmetry, it is a bit easier to see that it consists of heptagons. It is obtained by folding the heptagons according to the W fold. The author usually prefers to look at an odd-order symmetry axis, but for this model the 4-fold axis looks interesting as well. In that one you can recognise an eight-pointed star, while the model itself more seems to consist of disks.

Building Models

The models are built from a cardboard that is called Chromolux. This cardboard weights 250 g/m^2 and has one glossy side of a certain colour and one side that is plain white. The advantage of using this paper is that it does not look like paper at first glance and that the colours hardly fade in the sun. The disadvantage is

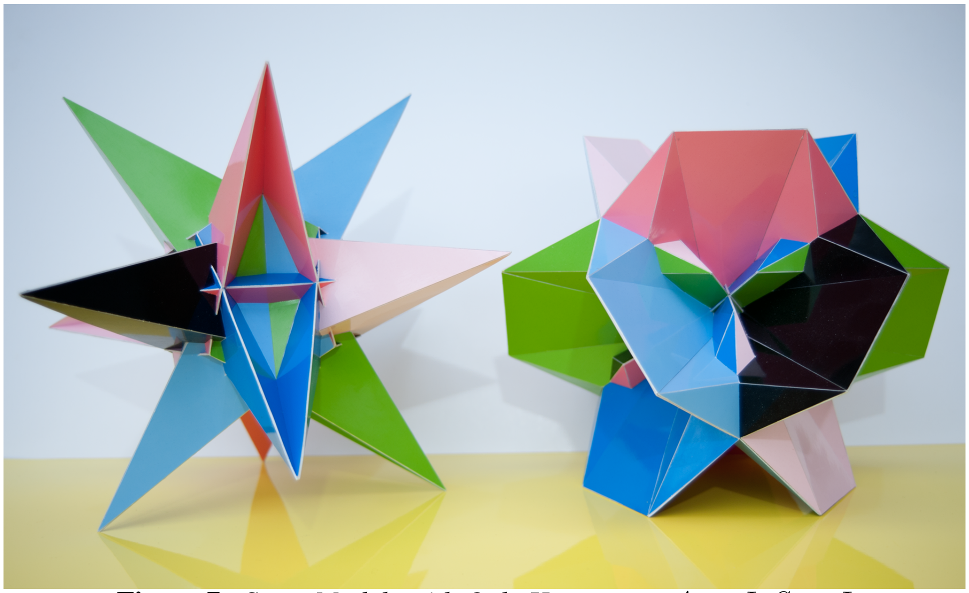

Figure 7 : *Some Models with Only Heptagons:* $A_4 \times I$*,* $S_4 \times I$

that it is very sensitive to glue, which sticks to the material immediately and removing it damages the glossy layer. Another disadvantage is that the material is rather thick.

To be able to build precise models the tabs are made out of normal paper instead of the cardboard. This is a bit of extra work, since after cutting out the pieces (using a standard pair of scissors) the author glues them onto a piece of paper to cut them out again. However, the extra work is rewarded; the result is very satisfying.

Recently the author bought a printer that can handle this kind of paper, but for the models that are shown here all pieces were still drawn by hand using a template, which is made of some thicker piece of cardboard.

The author uses different kinds of tweezers during the construction of the models: tweezers with broad flat areas to press together tabs, sharp pointed tweezers to work inside star arms, tweezers that enable one to hold tabs together around a corner, and crossed tweezers that can be used as a clip (the author likes to call these a pair of squeezers). Sometimes the author uses rubber bands to hold things together.

As an artist and a craftsman the author likes models that are small: preferably with a diameter smaller than 15 cm (or 6 in). The challenge here is, of course, that the material is rather thick and therefore not so suited for small models. Besides that you have less space to press and hold the tabs when finishing the models. So sometimes one has to decide to build a bigger model. Figure 7 shows some examples of models with a larger diameter. For the left one the smallest faces were left out. Although the author intended to add these in the end, the result without these was actually quite satisfying and the author did not think it would improve the model to add these. Moreover, the mathematical imperfection makes the model more interesting from an artistic point of view.

Conclusions and Further Work

It is the author's opinion that some models that were found are suited for artistic work, since they can surprise and intrigue the onlooker. Someone would react: "They look nice but just a bit random, or like any other model", but when they learn that all of these consist of *only* regular heptagons, they have a hard time believing it. The mathematician might say: "No, that is cheating, these are not really heptagons, they are triangles!" The artist would just smile and enjoy the controversy.

Even if the mathematician accepts this, the polyhedra that were presented here were generated numerically, using a precision of 13 decimals. From a mathematical point of view algebraic proof needs to be provided to make sure that these polyhedra really do consist of exact regular heptagons, though this is less important from an artistic side.

Some areas that can be searched for more are:

- Investigate all folds for the symmetries A_4, S_4 and A_5.
- Investigate other starting positions of the heptagon. E.g. instead of sharing an edge at an 2-fold axis the heptagons could share a vertex.
- Investigate more possibilities to fill the holes for A_4, S_4, and A_5.
- Search more combinations of different edge lengths.
- This paper investigates neither prism nor anti-prism symmetries, since the author is not much interested in these symmetries from an artistic point of view. F. Göbel described a simple example of a $D_{10}D_5$ consisting of 10 heptagons using a parallel fold.
- Investigate polyhedra consisting of other polygons that are allowed to be folded, e.g. pentagons or enneagons.

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