Abacaba! – Using a mathematical pattern to connect art, music, poetry and literature

Mike Naylor
Nasjonalt senter for matematikk i opplæringen
Norges teknisk-naturvitenskapelige universitet (NTNU)
7491 Trondheim Norway
Email: abacaba@gmail.org

Abstract

Any mathematical pattern can be used to create artwork. By exploring connections between patterns and the phenomena in which they occur, we can find not only surprising connections but also inspiration for creative works. We explore a simple binary fractal pattern and connect it to many areas of mathematics and the arts, including pictures, music, literature and poetry. Links to music, lesson plans and materials are provided for artists and teachers to explore this rich pattern further.

0. Introduction

The "abacaba" structure is a fractal pattern that shows up in an amazing variety of places. This paper explores many surprising ideas which all share this pattern, a path that will take us through geometry, number systems, art, music, poetry, literature, higher dimensions, and more. By finding connections and representing patterns in many ways, we can deepen our understanding of the beautiful mathematical ideas underlying our world and experiences and use these ideas to inspire artistic creations.

1. Ruler

The marks on an English ruler are the start of a fractal pattern. In the space of one inch, there's a big mark dividing the inch in half, two shorter marks dividing those halves in half, and so on. It's easy to imagine you could keep dividing the ruler again and again infinitely. If you did, you'd have a very interesting geometric object.

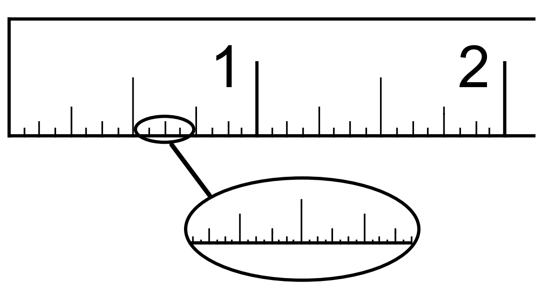


Figure 1: *The fractal pattern of an English ruler.*

If the shortest marks on the ruler are length 1, the next length 2, and so on, the pattern of the marks is 1213121412131215... This pattern shows up in all kinds of surprising places; let's give it a name.

2. The Name

Instead of using numbers to describe the length of branches or marks on the ruler, let's call the shortest lengths "a", the next longest "b," then "c," and so on. The pattern then becomes "Abacaba-Dabacaba!" This word sounds very much like the magician's phrase "abracadabra," and indeed there are seemingly magical properties about this pattern.

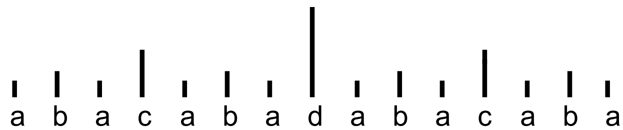


Figure 2: *Abacaba..., the name.*

To understand the pattern a little better and see how to continue it, let's see how this pattern grows. Start with an "**a**." To grow the pattern, add the next letter in the alphabet and then repeat everything that has gone before (which is just the letter "a" in this case.) The next step, then, is "**aba**."

Continue by adding the next letter, "c," and repeating the "aba" to make "**abacaba**." The fourth step adds the letter "d" and repeats the pattern to make "**abacabadabacaba**." The fifth step in the sequence is then "**abacabadabacabaeabacabadabacaba**" and the sixth step is the formidable "**abacabadabacabaeabacabadabacaba-fabacabadabacabaeabacabadabacaba**"!

It's fun to see how much you can say aloud. How long would it take to say the word all the way to "z"? This word is the basis for genie names in the children's story *Maggie and the Abacaba Genies* (see [3]). In the story, Maggie must call forth genies by saying these names, all the way to z!

This word was published in 2005 under the title "Abacaba unabridged, (volumes 1-4)" (see [4]). The 67 million character word, printed mostly in 4-point font, takes up over 1600 pages of A4 paper. This is not only a record for the longest published word, but is probably also a record for the most boring book in publishing history.

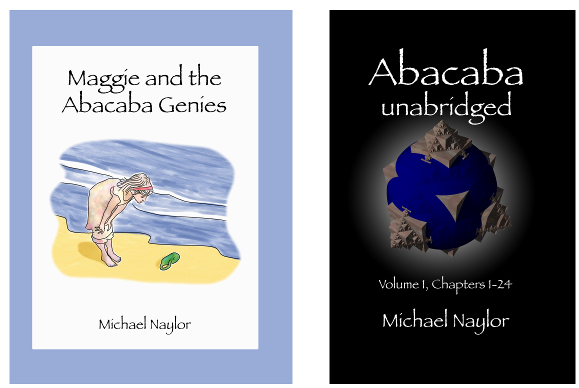


Figure 3: *Abacaba, the books.*

3. Trees

Another way to represent an abacaba pattern is with a binary branching tree. In this pattern, one object divides into two, then each of those divides into two, and so on, like the branches on this tree in Figure 4. It is not hard to imagine that the branching and doubling could be continued infinitely, and if we could only magnify our view enough times, we would see the same patterns continuing forever and ever. This same pattern is followed (in reverse) in a play-off schedule where teams or players are paired off with the winner of each round progressing to the next while the loser is eliminated. If we move from the top to the bottom of the playoff tree, we'll notice that the name closest to the top of the chart appears in column A, the next highest name is in column B, then column A, then C, then back to A, and so on: the abacaba pattern.

If we also think about the family tree containing our direct ancestors, we find the same pattern again. Abacaba is the pattern of everyone's lineage!

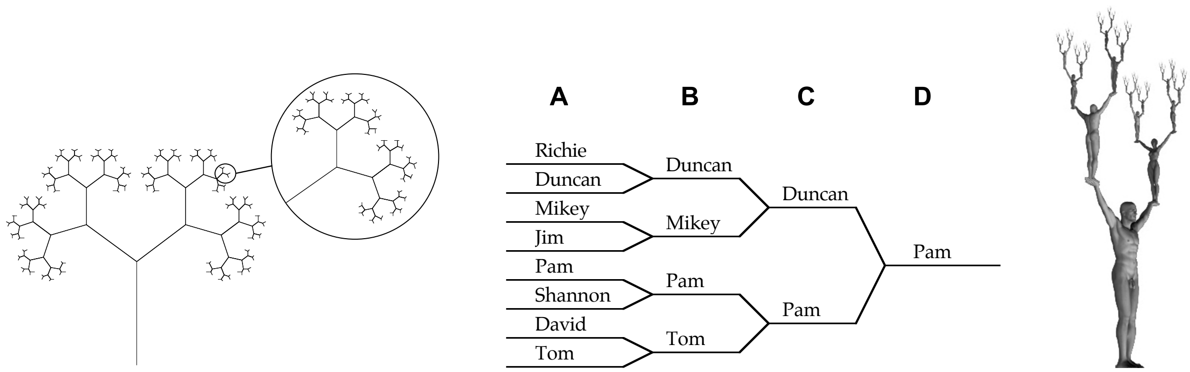


Figure 4: *Abacaba patterns in binary trees: a line fractal, a playoff tree, and a family tree.*

4. Fractal Connections

Abacabadabacaba is a fractal pattern – and it shows up in many other well-known fractal patterns. Fractals, like the ruler and the tree, have parts that are similar to the whole – if you zoom in and look closer, you'll see the patterns over and over. Here are three more famous fractals with abacaba patterns.

The Koch Curve is formed by replacing the center third of a line segment with two edges of an equilateral triangle. The Sierpinski Triangle (or Gasket) is formed by removing the center of a triangle and repeating. Cantor Dust is made by successively removing the center third of a line segment. Can you find abacaba patterns in each of these? Notice that the Sierpinski Triangle forms a ternary representation with the abacaba pattern flowing in three directions.

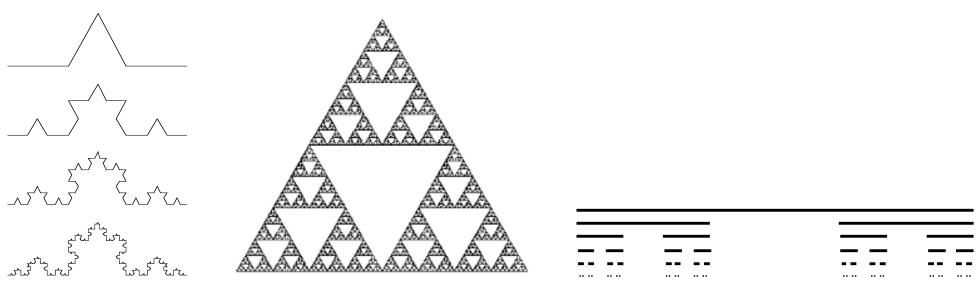


Figure 5: *Abacaba patterns can be found in the Koch curve, Sierpinski triangle, and Cantor's dust.*

5. Music

All those a's, b's, and c's sounds like they could represent a series of notes. Indeed, if abacabadabacaba is played on piano or other instrument, the result is a beautiful and haunting melody in the key of A minor. As one listens to the patterns, it becomes natural to shift the scale of listening to hear larger groupings of the pattern structure (for example, hearing the note pattern G-D-E-D-F-D-E-D with a 7-note ABACABA pattern between each of the stronger beats.) A copy may be downloaded from http://www.abacaba.org. There site also includes a fully orchestrated version in which abacaba patterns in instrumentation are played "upright" and "upside down", and abacaba patterns are present in percussion, volume, and stereo effects as well.

Figure 6: *Abacaba can be played as music.*

In Lerdahl and Jackendoff's G*enerative Theory of Tonal Music* (see [5]), the authors analyze accents in rhythms of both music and speech. Their theory posits an abacaba structure (or more correctly a *D*abacaba structure) as a naturally highly pleasing metrical structure. One example they offer is analysis of the first 9 bars of Mozart's G Minor Symphony, which they annotate with columns of dots to illustrate the prefered strength of emphasis on each pulse. In this example, each two-measure 16-pulse grouping has an EABACABADABACABA structure. They claim that the theme as a whole sounds richer and more "logical" because the rhythmic relationship is repeated at smaller and larger scales.

Figure 7: *Lerdahl and Jackendoff's analysis of accents in Mozart's G Minor Symphony.*

The abacaba pattern is also found in the musical structure of European music. Music from this period has six main stand-alone forms, one of which is the rondo form which features a recurring theme alernating with different sections, or episodes. A common symmetric rondo form is ABACABA, where the letters indicate placement of the musical phrases. Examples include the fourth movement of Brahms *Symphony No. 3*, the second movement of Beethhoven's *Symphony No. 3* (Eroica), and the final movement of Mozart's *Eine kleine Nachtmusik*.

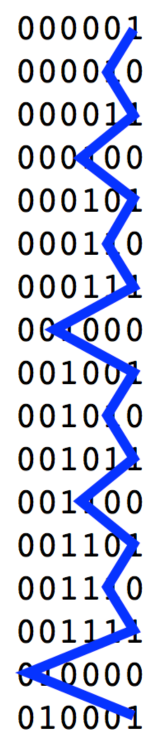


Figure 8: Binary *Abacaba.*

6. Binary and Gray Code

The binary number system is the simplest of the positional number systems. It contains layer upon layer of abacaba patterns. One such pattern can be found by looking at the position of the first digit 1 on the right side of each of the counting numbers. In 0001, the 1 is in position 1. In 0010 (two) the first 1 is in position 2. In 0011 (three) it is in position 1 again, and then in 0100 (four) the first 1 is in position 3. The sequence of positions is: 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, ...

Another abacaba pattern emerges if one examines the number of digits that need to be changed to switch from one number to the next in the binary counting sequence. From 0000 to 0001 requires changing one digit, from 0001 to 0010 requires two digits, and so on. The number of digit changes follows the sequence 1, 2, 1, 3, 1, 2, 1, 4, ...

Changing many digits at the same time in a count can be a problem for computers if the count is read in the middle of change; a small error can result in a big difference in the count. One solution used by data systems is to use Gray code. Gray code is a system of reorganizing the binary number sequence so that the change from one number to the next can be accomplished by changing only one digit. The count is resequenced in such a manner: 0000, 0001, 0011, 0010, 0110, 0111, 1111, ... The single digit that is switched follows an abacaba pattern: first digit 1 is changed, then digit 2, then digit 1, then 3, then 1, 2, 1, 4, and so on. The counting sequence then becomes 0, 1, 3, 2, 6, 7, 5, 4, 12, 13, 15, 14, 10, 11, 9, 8, 24, ... Because of the very high error-tolerance in binary systems, Gray code is also used in rotary encoder disks to electronically read angles using light sensors (see Figure 11).

In the representation below, we've drawn arcs to show the Gray code sequence on top of the standard integer sequence. The form should look familiar: it is yet another representation of the abacaba pattern!

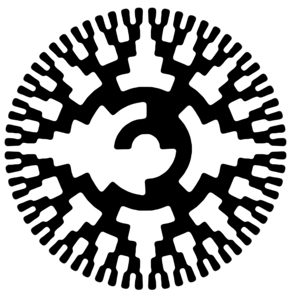


Figure 9: Gray code disk

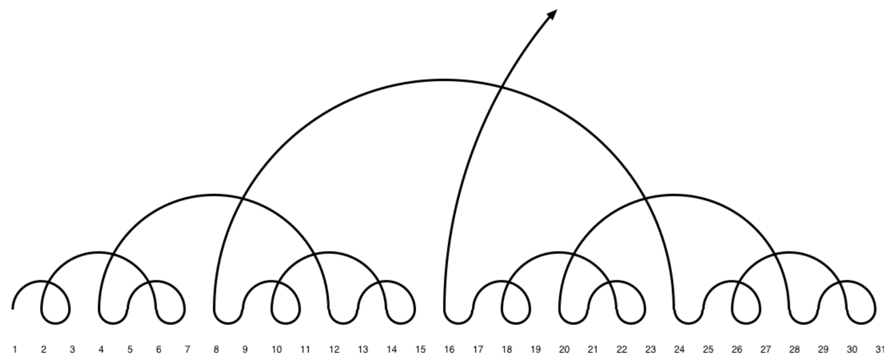


Figure 10: *Abacaba patterns in Gray code.*

7. Philosophy and Poetry

It's fun to think about how every decision we make leads us in a new direction, as if our lives are an infinite fractal tree. Here's a poem that reflects those decisions. It has the structure abacabadabacaba.

Decision Tree

And keep my conscience clear and bright.
I'll do what I know is right
Next time, who knows? I just might!
"I shouldn't do it," so I thought.
I guess I'm doomed to live this way.
Now the chance has slipped away
Maybe on some other day!
Should I do it? Should I not?
It wasn't worth it, I would say.
Now I'm full of guilty thoughts
But that's a tiny price to pay!
I went and did it anyway.
Tomorrow I will stay away.
I'll just hope I don't get caught
They didn't catch me yesterday!

Figure 11: *"Decision Tree" (see Naylor, [2])*

To read the poem, start with the largest line in the center and then decide whether to go up or down to the next largest line, and so on. The poem is loosely structured with the first line as the question, the second line as the decision, the third line the reaction to the decision, and the fourth line telling how this affects the future. There is an additional structure of moral (upwards) and immoral (downwards) thoughts. One striking result of this structure is that many of the possible paths through the poem produce somewhat depressing results, with the exceptions of the uppermost path and the lowermost path. In both of these paths, one is left with the feeling that the narrator is pleased with him- or herself. While this does not make any flattering statements about morality, it does suggest something about consistency. In life, as in mathematics, we are free to choose our own rules. The key to being satisfied is to follow these rules consistently.

8. Puzzles and Legends

A popular puzzle called "The Towers of Hanoi" asks players to move a stack of disks one at a time from one of three pegs to another peg. Only the top disc of a tower may be moved, and one may never place a larger disc on top of a smaller disc. The object is to transfer the entire tower to a different peg in the fewest number of moves. The key to solving this puzzle lies in the abacaba pattern. The pattern is also used to solve the linking rings puzzle known as "The Devil's Needle" or "The Chinese Rings", and it is also used in the puzzle "The Brain" manufactured by Mag-Nif.

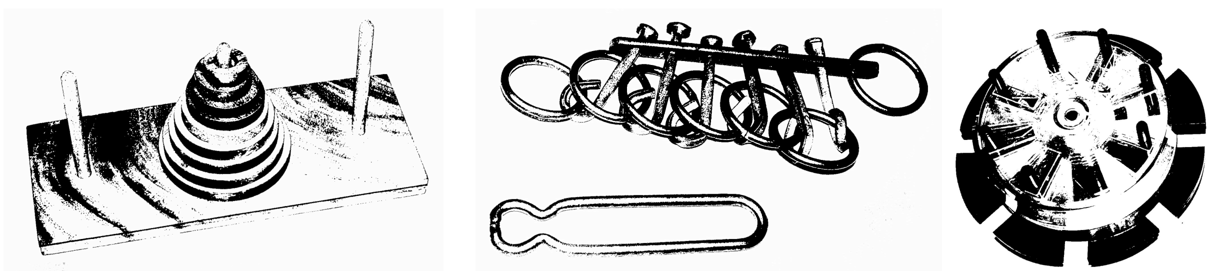


Figure 12: *The Towers of Hanoi, The Devil's Needle, and The Brain.*

There is a legend of a temple high upon a mountain where monks move golden disks on diamond spindles in this same pattern. When they have moved all 64, the universe will come to an end. Should we worry? (see [1]). It is easy to imagine the mountain is a tower of stone blocks arranged in a fractal staircase. One way it can be drawn is to construct the largest square possible inside a right isosceles triangle, and then continue drawing largest squares in the remaining triangles, and so on. The resulting staircase has the abacaba pattern.

9. Hyperspace

Navigating higher dimensions? Don't get lost! Call the left-right direction "a," the up-down direction "b," and the forward-back direction "c." Moving *a* will get you from one point to another when traveling on a 1-dimensional line segment. Moving *aba* will move you around the corners of a 2-dimensional square. Try moving *abacaba* on a 3-dimensional cube – did you visit all of its vertices?

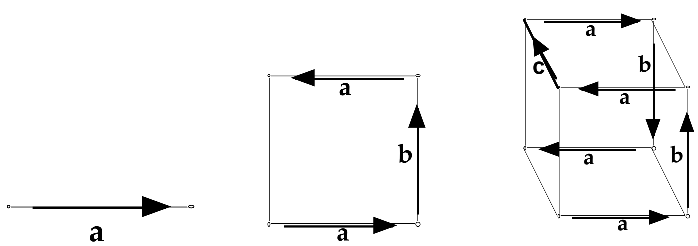


Figure 13: *Traveling on 1-, 2- and 3-dimensional figures.*

We can add another direction, the *here-there* direction, and call it *d*. To travel to all of the vertices of this 4-dimensional hypercube, just remember the magic word: Abacabadabacaba!

Figure 14 presents a 4D and a 5D hypercube. Are you up to the challenge?

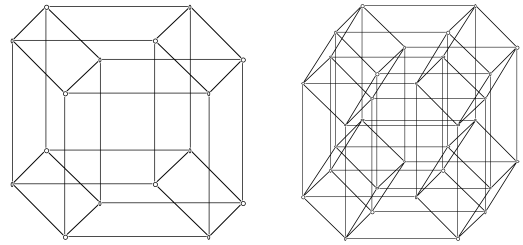


Figure 14: *Hyperspace challenge!*

10. The Complex Plane

The Mandelbrot set is a well-known fractal with infinite variety and complexity but generated by repeating very simple rules on the complex plane. Zooming in on the pattern reveals abacaba patterns seemingly everywhere you look. Shown here is a magnification of the "nose" of the figure, where it can be seen that the heights of the bulbs neatly follow an abacaba form. There are countless more abacaba patterns you can find in this remarkable structure.

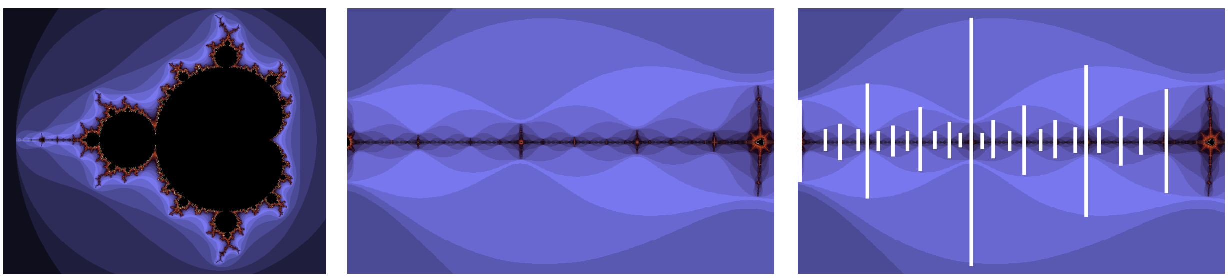


Figure 15: *Abacaba pattern in the Mandelbrot set.*

11. Other Areas of Mathematics

Abacaba patterns show up in many other places in mathematics as well. Here are a few:

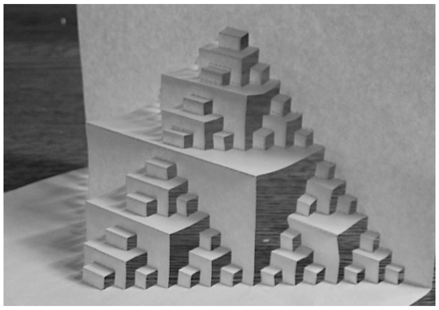
- ABACABA is used as a computer system back-up scheme. Data is backed up on drive A every other day, drive B every 4th day, and so on, resulting in a very good data spread and efficient use of available media.

- In the study of word structure, *abacaba*... is abelian square-free, meaning you will never find two adjacent blocks of letters of the same length which contain the same set of letters (in any order).

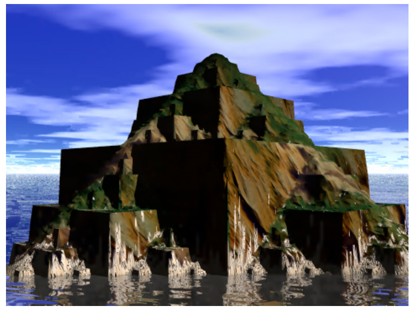
- In group theory, the elements of the Abelian group $Z_2 + Z_2 + Z_2$ are {abacaba = c, abacab = ac, abaca = abc, abac = bc, aba = b, ab = ab, a = a, e = e}.

12. Artwork

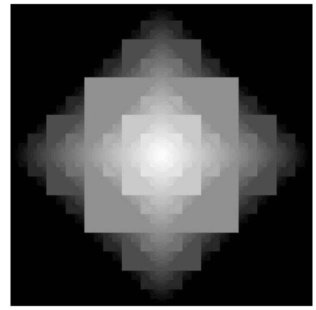
Like any mathematical pattern, abacaba can be used to structure all kinds of art. Representing a pattern in different ways can serve to inspire artistic creations. Here are a few last examples, created by the author with paper-cutting (1), Bryce (2), Mathematica (3), POV raytrace (4 and 5), and Appleworks (6).



Abacaba paper cutting

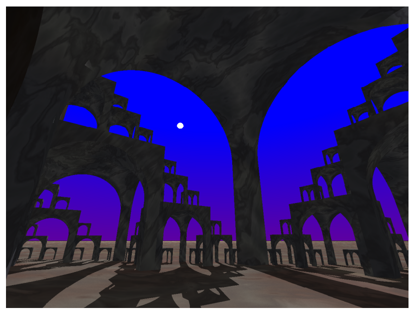


Abacaba island

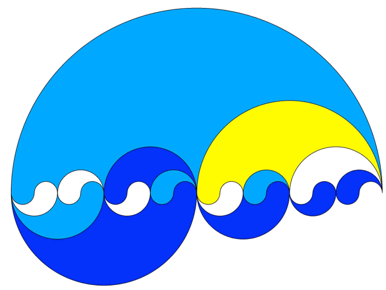


Abacaba op art

Abacaba caverns



Inside the Abacaba caverns



Abacaba yin yang wave

Figure 16: *Artwork based on abacaba patterns.*

13. Conclusion and Resources

We have explored many connections using just one pattern and represented this pattern in a variety of media: pictures, music, poetry and literature. Not only are there more connections to be discovered and represented with this pattern, but any mathematical pattern can be explored in similar ways. What kind of creative ways can we represent square numbers, prime numbers, the harmonic series, Pascal's triangle, or any of the countless interesting patterns and sequences in mathematics?

Many abacaba resources are available online at http://www.abacaba.org, including *Maggie and the Abacaba Genies*, classroom activities, worksheets, black-line masters and abacaba music.

References

[1] Gardner, M., *Hexaflexagons, Probability Paradoxes, and the Tower of Hanoi; Martin Gardner's First Book of Mathematical Puzzles and Games*, Simon and Shuster, New York, 1959.

[2] Naylor, M., *Decision Tree*, College Math Journal, 32, 3, May 2001.

[3] Naylor, M., *Maggie and the Abacaba Genies*, Lulu, Inc., Raleigh, 2005.

[4] Naylor, M., *Abacaba Unabridged (volumes 1-4)*, Lulu, Inc., Raleigh, 2005.

[5] Lerdahl, F. and Jackendoff, R., *A Generative Theory of Tonal Music*, MIT Press, 1996.