Human Geometry Workshop

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Abstract

What kind of geometry can we make with our bodies? In this 90-minute workshop we'll explore all kinds of fun ways to use our hands, arms, legs, and bodies to create fantastic shapes with ourselves and with a group. Frieze patterns, polygons, polyhedra, connected graphs, fractals and tessellations are a few of the exciting shapes we'll create. We'll present ideas, challenges, games, and problems to solve that will make you physically a part of mathematics! These ideas can be used in the math classroom to teach concepts and generate excitement for mathematics. Be prepared to get creative and bring home some great photographs.

Introduction

Geometry literally means "Earth Measure," and the earliest measuring tool was the human body. Hands and feet, paces and faces, polygons and legs – these are among the many body references used in measurement and geometry to this day [1]. By involving our bodies in mathematical reasoning, we can better understand geometric concepts and facilitate developing spatial sense by using what Seymour Papert called "body-syntonic reasoning" [2]. In this workshop we explore many ideas in which we can use our bodies to create mathematics.

Human Geometry Activities

Included here are outlines of some of the activities we will experiment with in this workshop. Structured activities will be presented and participants will also work in teams to create new forms and present their findings.

Handshakes. We are familiar with shaking hands when two people greet one another, but what about when three people greet one another? What kind of symmetries are involved in the action? What about with four or five people? [3]. With a partner, invent two or three new 2-person handshakes. Usually, handshakes are rotationally symmetric. Try to create a handshake with mirror symmetry, instead.

Now try a handshake with more than two people. In a small team of 3 to 6 people, come up with a group handshake. What symmetries are present? If all participants use the same hand (the right hand, traditionally), it's probably a rotational symmetry. One interesting handshake to try is to have all persons extend their right hands forward as if to give a traditional handshake, but instead have all of the fingertips meet at a point in the middle. All persons then curl their fingers at the same time, keeping the fingertips together. Another idea is for each person to grab the wrist of the person to the right. Countless formations are possible, some may only be possible with an even or with an odd number of people.

Figure 1: *A five-person fingertip-curl handshake.*

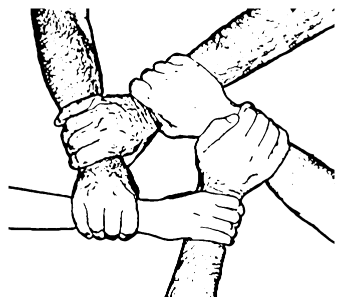

Figure 2: *Multi-person wrist-shake.*

Star polygons. When *n* people stand in a circle holding hands, and stretch out to be as far from each other as possible, we can think of the resulting shape as an *n*-sided polygon where each person is a vertex. People in a circle could also create star polygons, by holding hands with the person two (or more) places away from them, instead of their neighbor.

Try starting with groups of five. When five people stand in a circle and hold hands with the person two places away from each other, the result is a 5-pointed star. Can you untangle yourselves into a big open pentagon? Try different ways of weaving your arms and see if the result changes. Can you find all possible ways? It turns out that once you account for symmetry and rotations, there are four unique ways of weaving a 5-pointed star. Two can be untangled, and two can't.

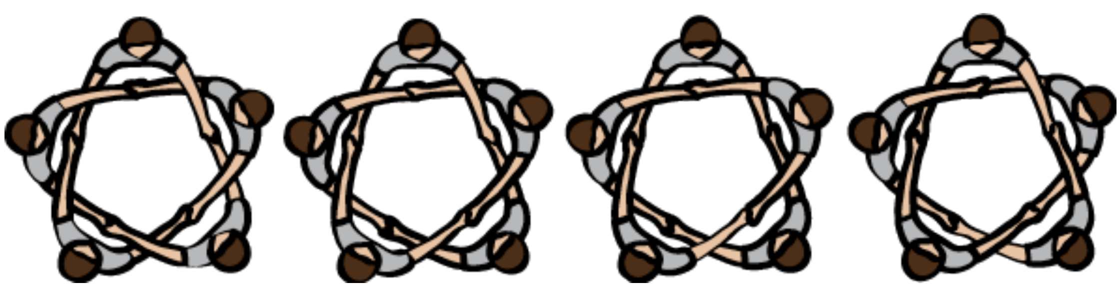

Figure 3: *Four unique ways of weaving a 5-pointed star.*

With six people, holding hands with the person two away gives another kind of star, made up of two separate triangles. Can they be separated, or are they linked together? For n people in a circle holding hands two people away, how can you tell just by looking at the weaving pattern if you can untangle or not? Think about the kinds of "operations" you do in order to physically untangle, and see if you can come up with a theory.

To make a proof, a first step might be to figure out a good way to notate how the arms are weaving. One way might be to follow the weave around and write down the pattern of overs and unders, or you might go around the circle of people and write down whether their right arm is going over or under. Any method of notation that gets down all the information in a series of letters and numbers is good, and another thing to look for is that it contains only the necessary information and no redundancies (for example, if each person went around saying whether their left arm is over or under, and then whether their right arm is over or under, you would have twice as much information as you need, as one person's left arm weaving is dependent on their left neighbor's right arm, and vice versa).

Within your notation, try to "untangle" the star by figuring out what terms can cancel out, drawing from physical experience. Figure 4 shows a couple of different ways to notate a 5-pointed star, and how a person with both their right and left hands woven under can step beneath the weavings to simplify the tangle.

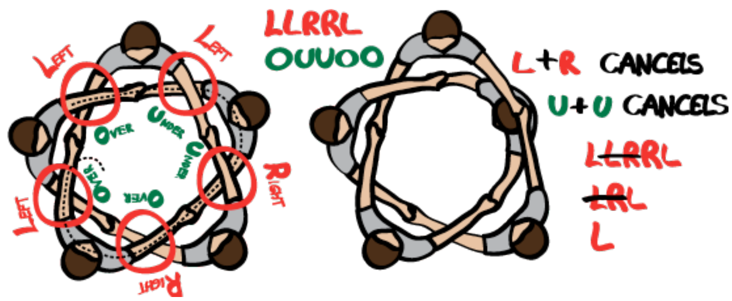

Figure 4: *Two ways of notating the weaving of a star, and canceling out terms to untangle it.*

This exercise, in addition to being an example of simple mathematical proof, also shows the usefulness and arbitrariness of notation. Notation is a useful tool, and it's okay to make it up if you need to!

With larger values of *n*, it's possible to explore other kinds of stars, where group members hold hands with the person three or more places away. How do you know whether you'll get a continuous loop or separate polygons?

See what else happens. In the process of untangling, a group might end up facing out instead of in. Does this depend on how they wove arms, or could it happen to anyone? Could you ever have just one person end up facing outward?

Polygons. In small groups, explore different ways of making polygons with your bodies. Using fingers, arms, legs, and/or your entire bodies, figure out how to make polygons both irregular and regular, symmetric and asymmetric. Polygons may be made just using hands, using arms and/or legs when standing up, or even by laying of the floor and using whole bodies. Have each group come up with some formations they especially like and share with the others. Possibly you can make more than one polygon at once. Figure 5, for example, shows a way of making a triangle inside a hexagon, and a rectangle inside another rectangle. Then have the groups share their polygonal discoveries!

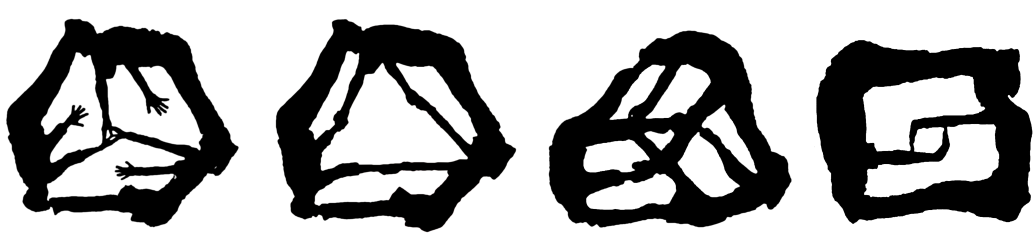

Figure 5: *Multi-person polygons made on the floor.*

Frieze patterns. A Frieze pattern is a repeating pattern that might have other symmetries as well: mirror lines, 180° rotations, and glide reflections, sometimes in combinations. For example, a bunch of people standing straight in line creates a shape with two symmetries: repetition symmetry (roughly), and mirror symmetry (also roughly, though less so). If everyone faces the front of the line, the mirror symmetry goes straight down the center of the line (as seen from above). If everyone were to turn 90° to the right, the line would have a different set of mirror symmetry. There would be lines down the middle of each person, and also lines between each person. These are two different Frieze patterns. Then, if every other person were to turn around, you'd get yet another Frieze pattern, with a set of mirror lines down each person and points of rotation between each person.

Standing straight is boring, though. Have someone change their shape slightly, by holding out an arm or leg however they choose. The next person in line then has to choose whether to copy them directly, or mirror their action. To continue the pattern, either everyone has to mirror, or no one mirrors. Complete the pattern and figure out what symmetries it has.

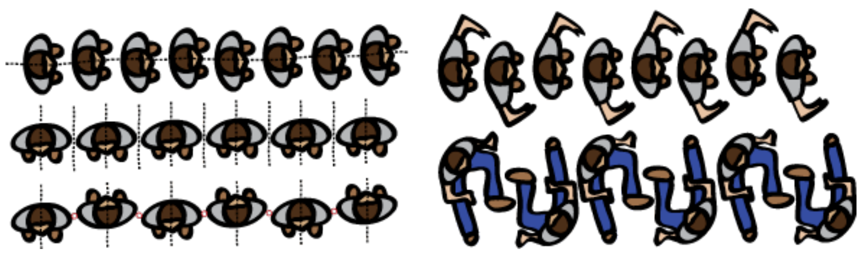

Figure 6: *Five different human Frieze patterns.*

Once you've got the hang of it, try building frieze patterns from scratch. Start by having one person choose a shape to be in, and have another person copy that shape next to them. They can copy it exactly, or mirror the first person, or copy it but rotate 180°. The next person's job is to figure out how to continue the pattern. There might be more than one way that still creates a Frieze pattern! Have more people join, one by one, and see what happens.

There's seven types of Frieze pattern in all. For an advanced challenge, starting with the same shape, build all seven patterns one by one.

Tessellations. We can analyze tessellation patterns to divide edges into groups that resemble human forms. Using arms, legs, and bodies as edges we can create some of these big tessellating patterns.

Square patterns are the easiest to start with, and there are many many possibilities. Three examples are shown in Figure 7.

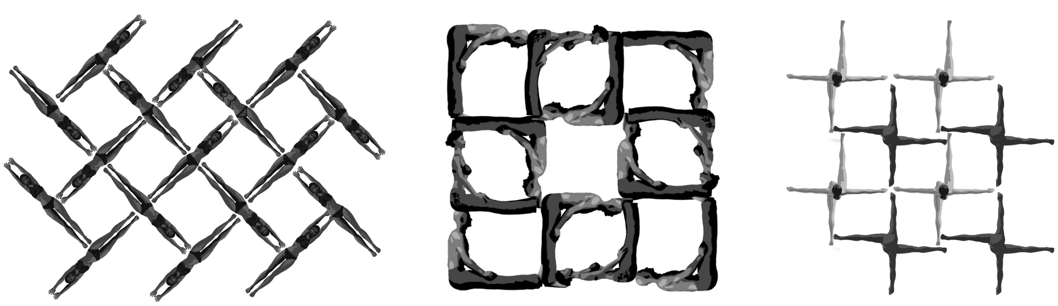

Figure 7: *Square tessellations with people contributing 1/2, 2 and 4 edges.*

For more complicated patterns, a useful technique is to start with a line drawing of a tessellation and identify clusters of edges that could be represented by a human form. Finding a pleasing arrangement that works involves a mixture of art, logic and visualization. In this example, we start with a 6.3.3.3.3 semi-regular tessellation and sketch a stick-figure with two legs, facing outwards from a hexagon. Applying rotational symmetry results in a promising Star of David shape which could be tessellated but would have some overlapping edges. We could use two different poses for our models, but instead we ask everyone to keep their feet together and reach out a forearm to occupy half of a shared edge. The result is shown in Figure 8.

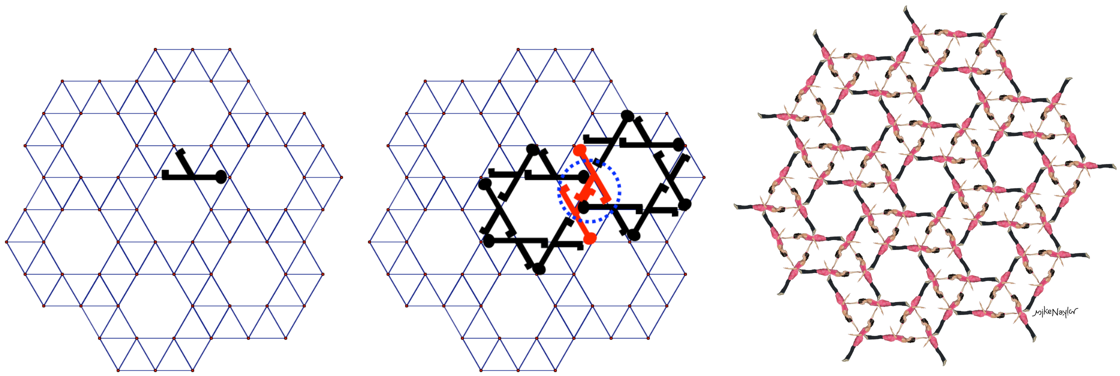

Figure 8: *Planning a tessellation with stick-figures – a false start and a final result.*

Fibonacci family trees. Honeybees have an interesting family tree. Unfertilized bee eggs produce male drones, while fertilized bee eggs produce female worker bees. These bees are sterile, unless they are fed royal jelly, and then they develop into queen bees who are capable of laying eggs. Thus, male bees have only one parent and female bees have two parents, resulting in an unusual family tree and a surprise occurrence of Fibonacci numbers. Making a bee family tree with a class of students is a fun way to introduce this mystery.

Choose a male to be a drone bee, and have him stand at one end of the room. He should choose a female in the room to represent his mother, and she should near him. She should then choose a male and a female as her parents, who then stand in group third in line. These two should then pick their parents, and the next group will then have 3 members: one male and two female. Continue until there are not enough people to make another generation. Remember to keep generation groups separate so the groups don't become mixed.

Going back to our first bee, count the number of parents, grandparent, great-grandparent, great-great-grandparents, and so on. The sequence should be 1, 1, 2, 3, 5, 8, 13, ...

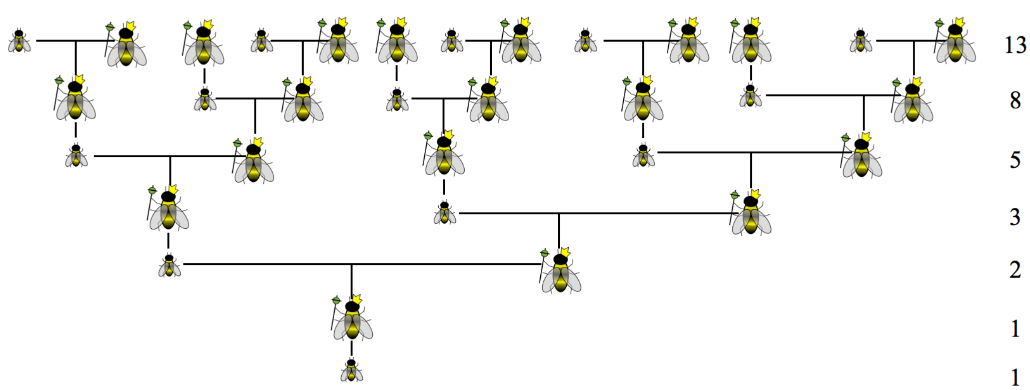

Figure 9: *The honeybee family tree is a fun structure to model with a class!*

Hand polyhedra: Making polyhedra with our own hands, either by working with a partner or working with a small group, is satisfying and helps us understand relationships within polyhedra ([1] and [3]). Here are several ideas for hand polyhedra – what kind of 3d symmetries are involved in these creations, and what kind of polyhedra can we invent ourselves? Many of these wonderful creations come from "Math Dance with Dr. Schaffer and Mr. Stern".

a. Partners hand cube: Work with a partner. Place the tips of your thumbs together and the tips of your middle fingers together to make a square. Rotate your hands so that you are pointing your index fingers towards your partner and the square is facing him or her. Your partner should do the same. Now one of you should rotate your square 90° so that your index fingers are in the up-down position instead of the left-right position. Bring your squares together so that your index fingers meet the free corners of your partners' square; your partner does the same. You've made a cube! [3] Follow this with the...

b. Partners hand tetrahedron: Work with a partner. Place the tips of your thumbs together and extend your index and middle fingers towards your partner like you're pointing at them with two pairs of scissors. Your partner does the same. Your four extended fingertips should be in the shape of a square. Bring your square arrangement of fingertips towards your partner's square arrangement of fingertips. One of you rotates your square 90°, and then you touch your fingertips together, maintaining the square arrangement. You've made a tetrahedron! [3]

c.Transforming challenge! Can you and your partner figure out how to transform the cube into the tetrahedron and back again? [3]

d. Tetrahedron. [3] A very nice tetrahedron is possible with your own two hands. Each hand will make 3 of the 6 edges, the three edges are comprised of the thumb, the forefinger, and the stretch of hand between the thumb and the forefinger. Bend your forefinger at the second knuckle to form a 60° angle, and lift your thumb to the side of your hand to make another 60° angle. Do the same with your other hand. Can you figure out how to assemble these two pieces to make a tetrahedron? Keeping your hands in the same shapes, can you figure out a different way to put the two halves together to make an identical, but mirrored, tetrahedron?

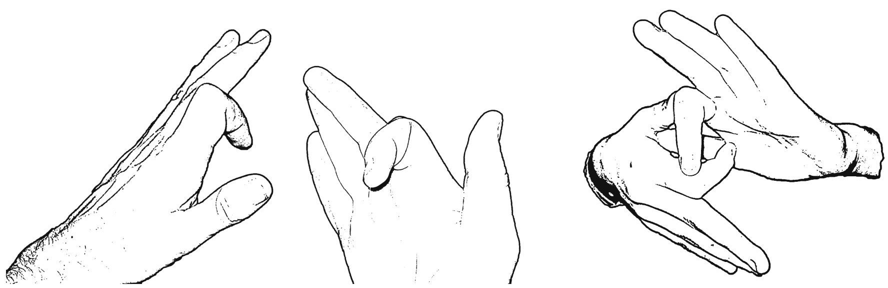

Figure 10: *A two-hand tetrahedron.*

e. Tiny tetrahedron. Touch your thumb and index finger together on both hands, then press both pairs of fingertips together with one pair rotated 90°. The empty space between your fingertips is in the shape of a tiny tetrahedron!

f. Other hand polyhedra.

- The thumb, index, and middle fingers can point in three orthogonal directions like the corner of a cube. Can you make a cube with a 4 pairs of hands?
- Each hand has 5 fingers, and each vertex of an icosahedron is the meeting place for 5 edges. Since an icosahedron has 12 vertices, can you make a complete icosahedron with 6 pairs of hands?

Human polyhedra. Making polyhedra in a group using your whole body is a fun challenge. Here are a few ideas:

a. 3-person tetrahedra. There's lots of ways to make tetrahedra with a team of three. Each person is responsible for 2 edges, or edges can be shared.

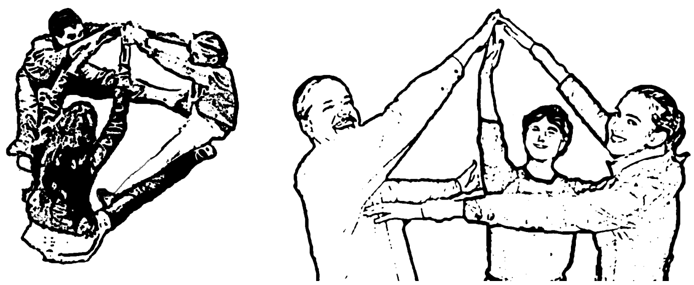

Figure 11: *Three-person tetrahedra.*

b. 2 person cubes. With each person responsible for 6 edges, making a two-person cube can be a challenge. Here are two examples, one with 2 missing edges which nonetheless gives an excellent illusion of a cube.

Figure 12: *Two-person cubes.*

c. 4-person cubes. There's plenty of ways to make cubes with 4 people. One of the easiest ways to sit in a square and make the bottom edges with legs and the top edges with arms. Arms and legs can overlap or be bent back so that knees and elbows meet. Bodies can also make three edges of a cube with the two legs split at 90° as two of the edges and the torso as the third edge. One somewhat acrobatic way is shown below – if this cube were "rolled" 90° it could be made without needing headstands.

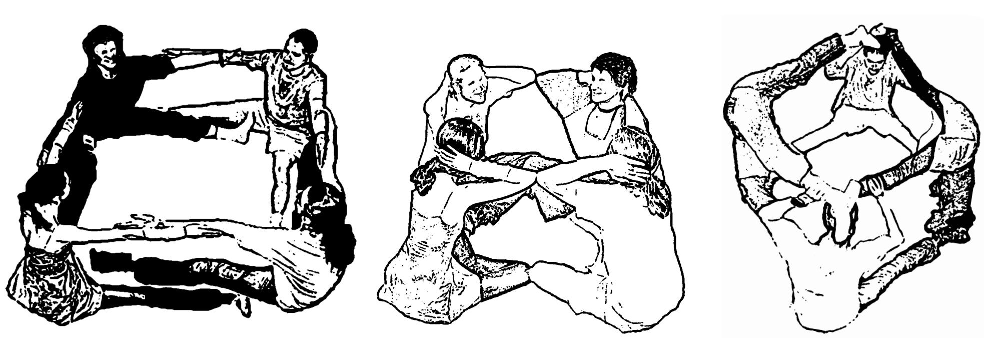

Figure 13: *Four-person cubes.*

d. 6-person cube. Bodies bent 90° at the waist make two edges of a cube. Can you make the complete cube with 6 people?

e. Other polyhedra: In a group, come up with ways to make your own polyhedra. Tetrahedra and cubes are probably the easiest, but the octahedron could be an interesting challenge. It is useful to remember that tetrahedra have 6 edges and 4 vertices, cubes have 12 edges and 8 vertices where 3 edges meet, while octahedra have 12 edges and 6 vertices where 4 edges meet.

Fractals. The Koch curve, Sierpinski triangle, and binary trees are a few of the fractals that can be created with a group. How many people do we need for each fractal? Are there several ways to create the same form? Are 3d fractals possible? Figure 14 shows a couple of ideas.

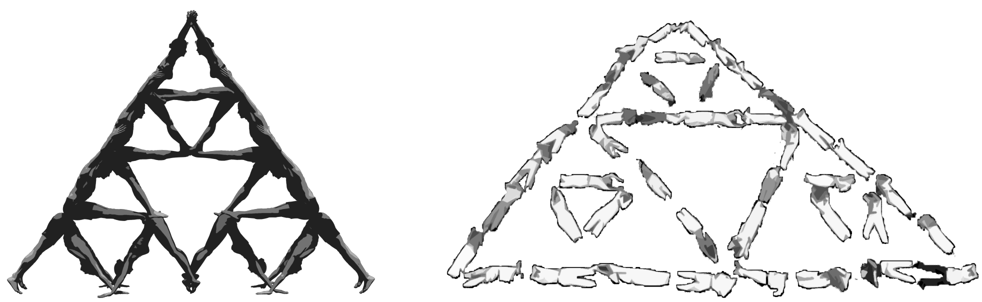

Figure 14: *Ideas for human Sierpinski triangles.*

Thanks

Special thanks to Robert Bosch, Erik Demaine, Eva Knoll, and Karl Schaffer for their contributions to the project.

References

[1] Naylor, M., *Naked Geometry,* Lulu, Inc., Raleigh, 2005.

[2] Papert, S., *The Children's Machine: Rethinking School in the Age of the Computer*, Basic Books, New York,1992.

[3] Schaffer, K., Stern, E., and Kim, S., *Math Dance*, Karl Schaffer, Erik Stern, Scott Kim, Santa Cruz, 2001.