Models of Locally Regular Heptagonal Dodecahedra

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Abstract

We present models of polyhedra built from 12 heptagons meeting three per vertex. Unlike the analogous case with 12 pentagons, where there is a single unique combinatorial structure, there are six combinatorially distinct ways to combine 12 heptagons, meeting three per vertex, into a (possibly self-intersecting) polyhedron. We identified realizable (non-self-intersecting) examples for five of the six possible structures, and fabricated physical models of them. They all necessarily have genus 2 (topologically equivalent to a 2-holed donut), and they appear in a variety of aesthetically pleasing symmetries. These models demonstrate a form of art emerging from mathematics.

Six years ago, J.M. Wills [4] proved that a trivalent heptagonal dodecahedron could be realized, and more recently [5], he constructed an explicit example. His example was combinatorially different from the one used in his earlier proof, revealing the evident fact that there were at least two combinatorially distinct ways to combine 12 heptagons, meeting three per vertex, into a polyhedron.

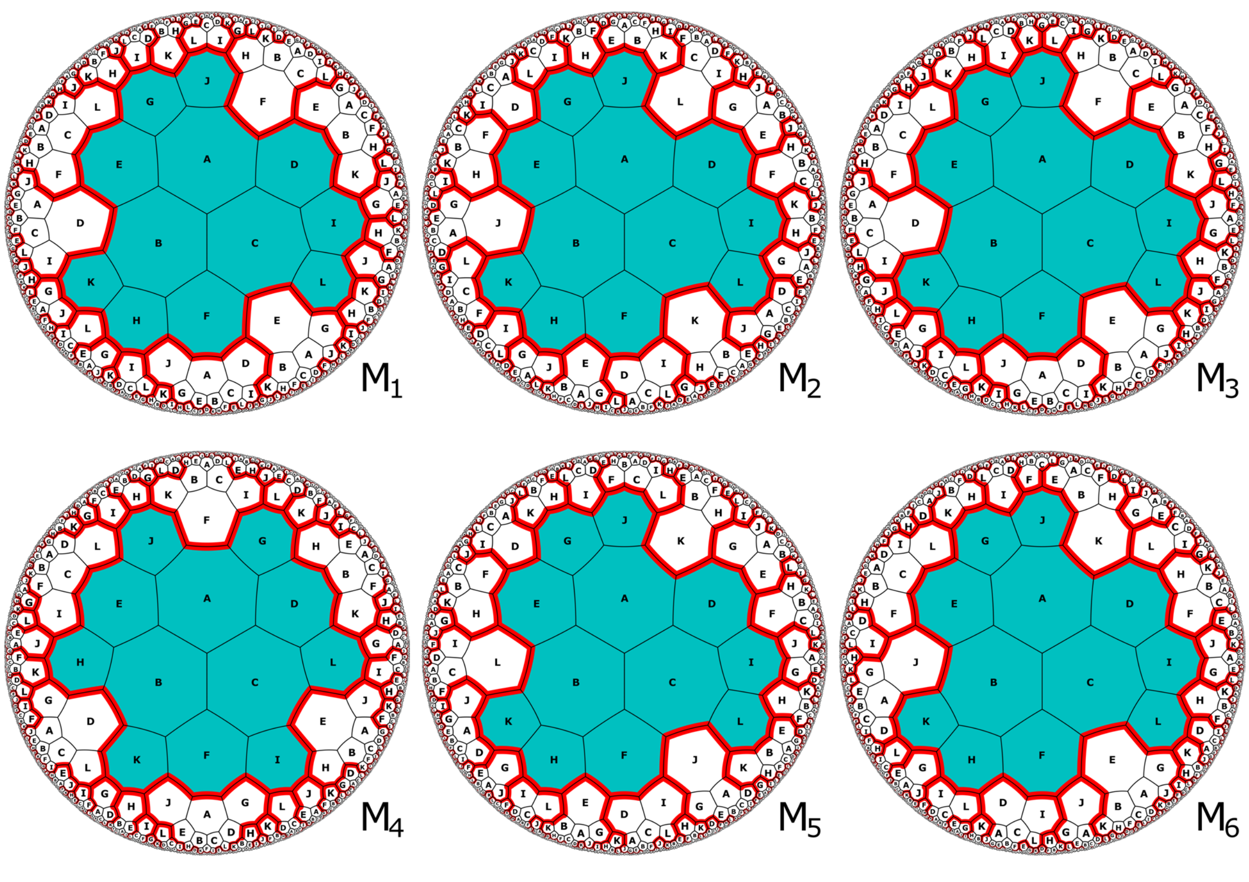

Figure 1: *The six combinatorially distinct structures for a trivalent heptagonal dodecahedron.*

It turns out that there are exactly six combinatorially distinct ways to combine 12 heptagons, meeting three per vertex, into a (possibly self-intersecting) polyhedron [1]. Maps for these six structures are shown in Figure 1. See [3] for a thorough description. In this short paper, we present polyhedral realizations without self-intersections for five of these maps (M_1, M_2, M_3, M_5, and M_6). They were found using the same algorithm that was used for finding realizations of the all-heptagon Klein map [2], except that the symmetry of the expected polyhedron in each case was chosen to be consistent with the associated map's automorphism group [3]. The polyhedron described by Wills in his 2004 existence proof is a realization of M_1, and the one he constructed explicitly in 2008 is a realization of M_2. Any realization of M_4, if it exists at all, has only the trivial symmetry [3]. Figures 2 through 7 show stereo pairs and nets for these polyhedra, and Figure 8 shows some physical models. Examples were selected with qualities such as 90-degree angles, parallel faces, and balanced edge lengths.

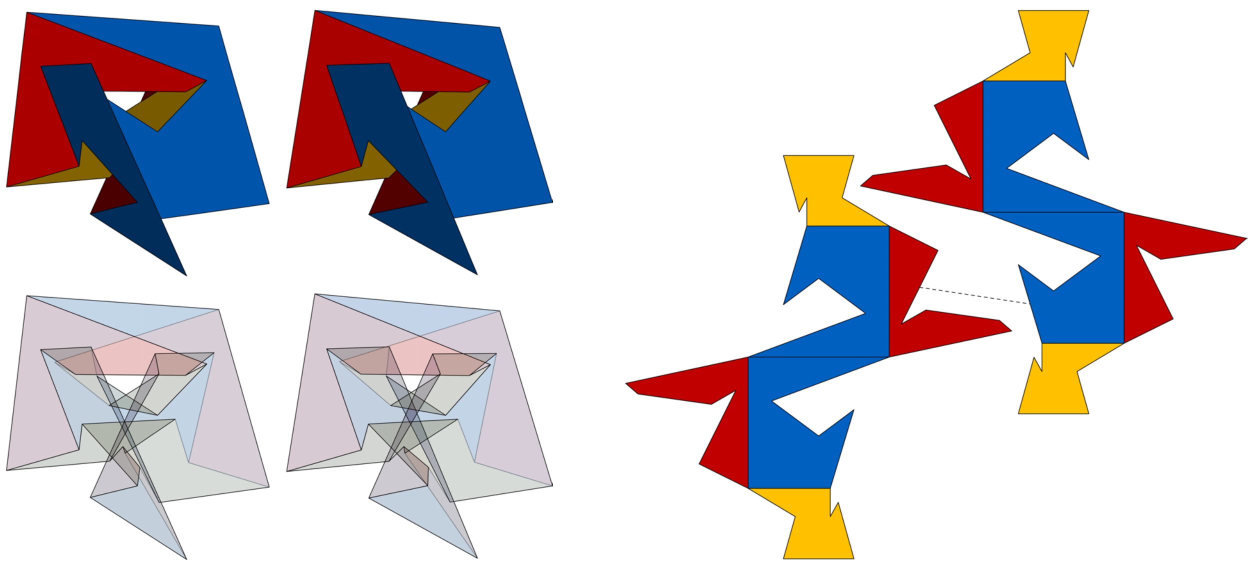

Figure 2: *Two stereo pairs and a net for a polyhedral realization of M_1 with S4 symmetry.*

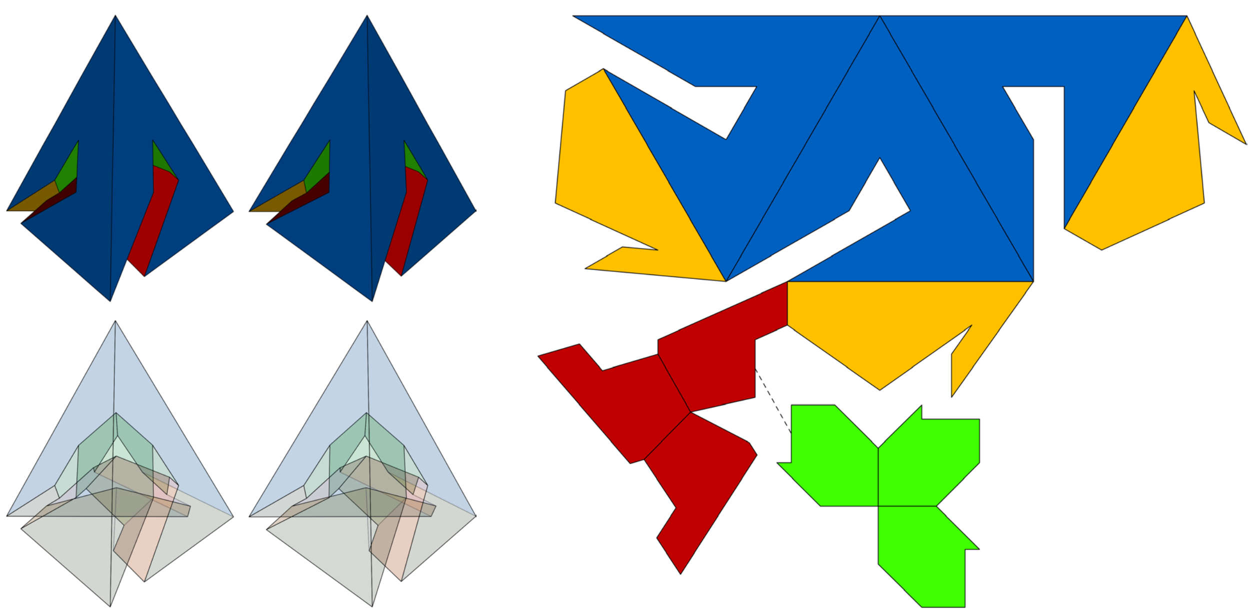

Figure 3: *Two stereo pairs and a net for a polyhedral realization of M_2 with C3 symmetry.*

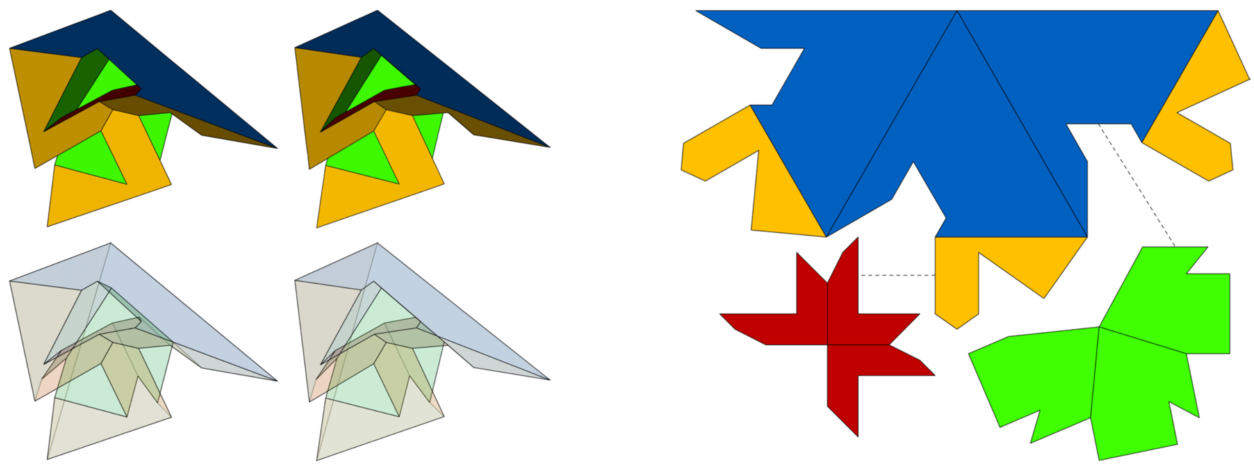

Figure 4: *Two stereo pairs and a net for a polyhedral realization of M_2 with C3 symmetry.*

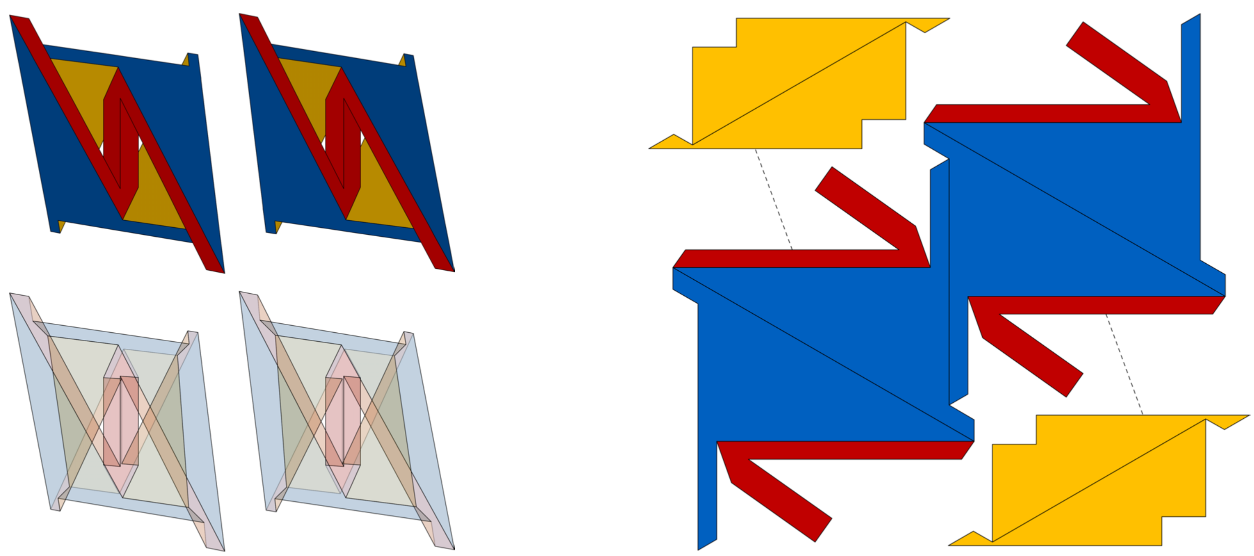

Figure 5: *Two stereo pairs and a net for a polyhedral realization of M_3 with D2 symmetry.*

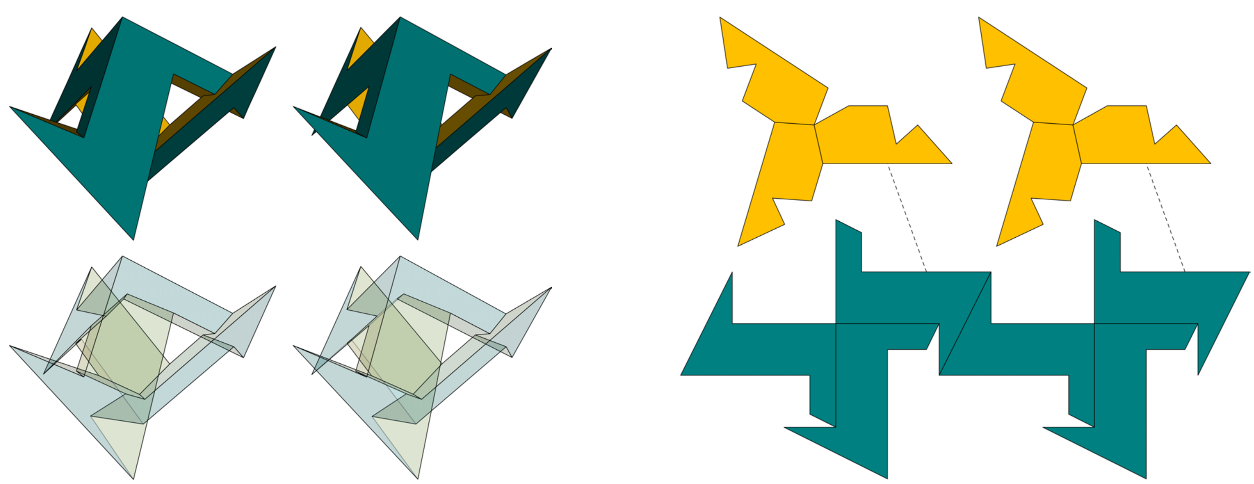

Figure 6: *Two stereo pairs and a net for a polyhedral realization of M_5 with D3 symmetry.*

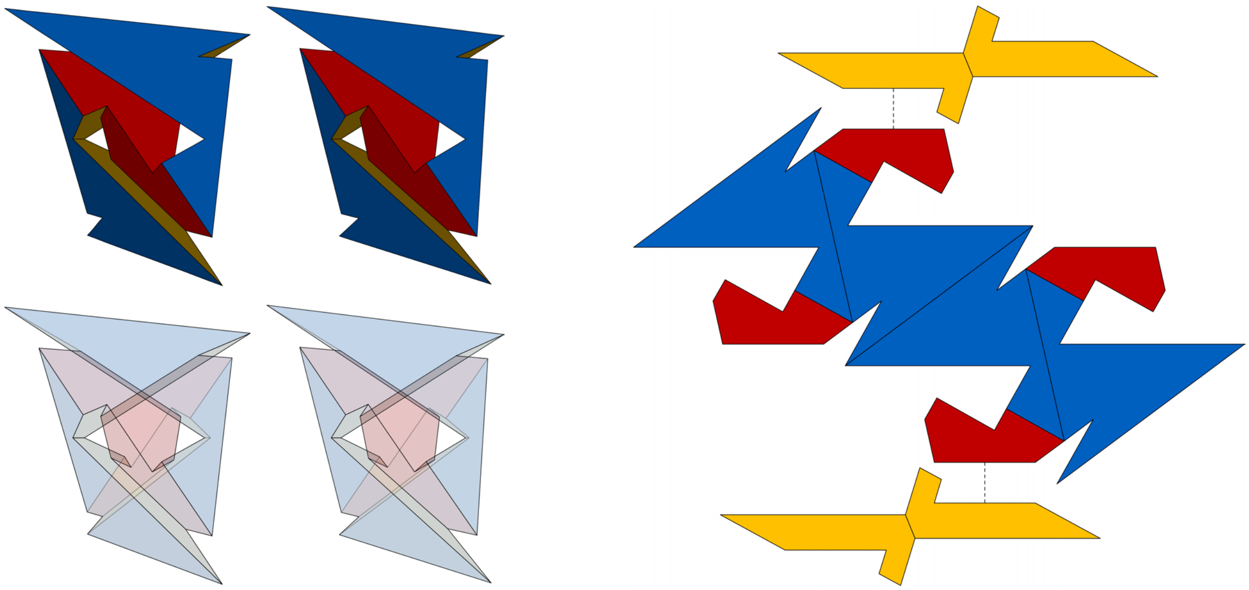

Figure 7: *Two stereo pairs and a net for a polyhedral realization of M_6 with D2 symmetry.*

We fabricated stereolithography models at scales of roughly 3.5 inches in diameter, and paper models at scales of roughly 7 inches in diameter. Besides being significant as the simplest possible locally regular genus-2 polyhedra, these models also serve as useful representations of some simple point group symmetries, not to mention their aesthetic appeal as art objects. The M_1 polyhedron, in particular, is particularly interesting for its 4-fold rotoreflection (S4) symmetry, which is not often encountered.

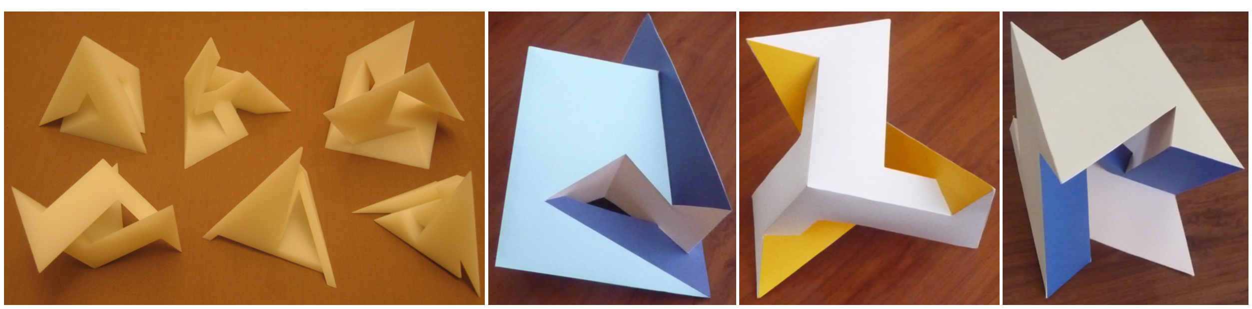

Figure 8: *Stereolithography models ([M_2, M_2, M_1] [M_5, M_3, M_6]) and paper models (M_1, M_5, M_6).*

References

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[2] D. I. McCooey, *A non-self-intersecting polyhedral realization of the all-heptagon Klein map,* Symmetry: Culture and Science, 20(1-4), pp. 247-268. 2009.

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