Bourgoin's 14-Pointed Star Polygon Designs

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Abstract

In this paper we explore a contemporary conceptualization and creation of four, historical 14-pointed star polygon designs illustrated and analyzed in Plates 164 – 167 of Bourgoin's *Arabic Geometrical Pattern and Design* [1]. We seek to determine how the original designer of these patterns may have determined, without mensuration, the proportion and placement of the star polygons comprising each pattern. We will do this by proposing a plausible Euclidean "point-joining" compass-and-straightedge reconstruction for each.

Introduction

The tiling of surfaces with elaborate, infinitely-repeating geometric designs is one of the distinguishing characteristics of Islamic art. Most of these geometric patterns are highly symmetric and may be generated by inscribing regular polygons within circles to divide the space evenly, yielding regular and star-shaped polygons. The most commonly-occurring geometric Islamic designs are based on n-gons (polygons with n sides), where n = 3, 4, 5, 6, 8, 10, 12, and 16. These regular n-gons are also *constructible* in the Euclidean sense; that is, they may be created using only a compass to make circles and a straightedge to connect points of intersection between "path objects" such as segments and circular arcs.

For n = 7, 9, 11, 13, 14, 18…, the regular n-gons (and likewise, the corresponding regular n-star polygons) may be constructed only approximately using these tools. Of these *non-constructible* star polygons, we will explore the conceptualization and creation of four 14-pointed star polygon designs found in Plates 164 – 167 of Bourgoin's *Arabic Geometrical Pattern and Design* [1], a rich source of 190 Islamic patterns, first published in 1879 and based upon drawings, with analyses, of Islamic monuments in Cairo and Damascus. We specifically seek to answer the question, "How did the original designer of these patterns determine, without mensuration, the proportion and placement of the star polygons comprising each design?" In addition, we propose a plausible Euclidean "point-joining" compass-and-straightedge reconstruction for each, using the *Geometer's Sketchpad* software program [2], the electronic equivalent of the compass and straightedge.

Construction of a "Nearly Regular" 7-gon and 14-gon, and their Inscribed Star Polygons

In this section, we outline a method to construct star polygons that occur in Bourgoin's plates 164 - 167. A straightforward technique for creating star polygons is to initially construct a regular p-gon inscribed in a circle and then draw in the corresponding regular p-pointed star by methodically joining successive qth vertices of the p-gon with line segments (diagonals) or by similarly joining midpoints of the p-gon's edges. A figure formed in this way is mathematically designated as a $\{p/q\}$ star polygon, where p and q are positive integers that are also relatively prime, with $q < p/2$.

For example, to create a "nearly regular" seven-pointed {7/2} star polygon, we construct an approximately regular heptagon and then connect every other vertex with line segments. A very good approximate heptagon and its inscribed 7-pointed star polygon, shown in **Figure 1a,** was constructed using a method attributed to Abu'l-Wafa' al-Buzjani in a 10th Century treatise entitled *Kitāb fimāyahtāju ilayhi al-sāni'min a'māl al-handasa* ("About that which the artisan needs to know of geometric constructions"[3]) as described by Sarhangi in [4]. Other approximate constructions of heptagons are discussed by Berggren [5], Hogendijk [6], and McBurney [7].

From an inscribed heptagon, a 14-gon may be generated by constructing lines through the vertices of the heptagon and the midpoints of the corresponding opposite line segments. The points where these lines intersect the circle divide the circle into 14 approximately congruent arcs and also form the vertices of the 14-gon, as shown in **Figure 1b**. A {14/5} star polygon may be created by joining with line segments every fifth vertex of the fourteen found on the circumference of the circle (**Figure 1c**). By erasing some segments and highlighting others, we generate the more decorative star polygon image in **Figure 1d**.

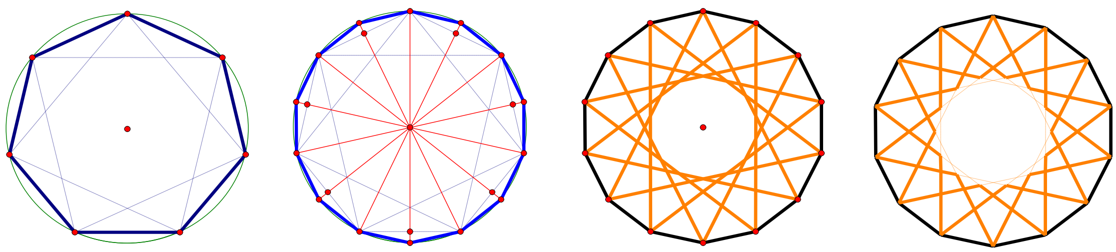

Figure 1a. **Figure 1b.** **Figure 1c.** **Figure 1d.**

Construction of the Fourteen-Pointed {14/6} Star Polygon

A {14/6} star polygon forms the basis for the 14-pointed star patterns found in Bourgoin's Plates 164 – 167. To produce this star polygon, we start with a circle that has 14 approximately congruent arcs (**Figure 2a)**, connect every third point with line segments, then mark the innermost 14 points of intersection, as shown in **Figure 2b**. We repeat this procedure with the 14 marked points of intersection as shown in **Figure 2c**; this produces a {14/3} star polygon within a {14/3} star polygon. Finally, to generate the {14/6} star polygon of interest, we connect every sixth point of intersection marked in the previous step, to produce **Figure 2d**.

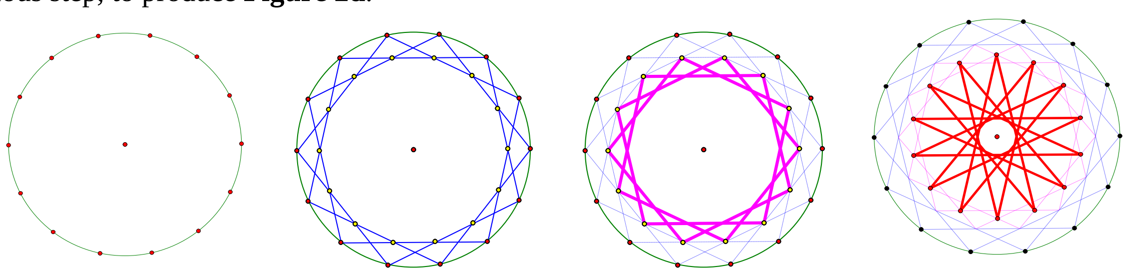

Figure 2a. **Figure 2b.** **Figure 2c.** **Figure 2d.**

Erasing the segments generated in the second step **(Figure 2c)** and highlighting the appropriate segments that form the 14-pointed star yields the image in **Figure 3a**. Bourgoin's star designs have additional embellishments surrounding the star. To produce these, we extend the sides of the star polygon until they intersect the line segments generated in the first step, thereby creating additional points of intersection from which the hexagonal "arrow" shapes may be drawn. Two of these chords are shown in **Figure 3b,** along with the arrow they determine at the top of the 14-pointed star.

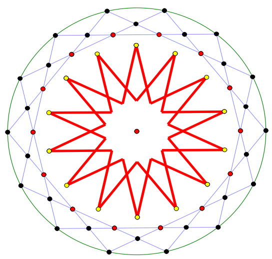

Figure 3a.

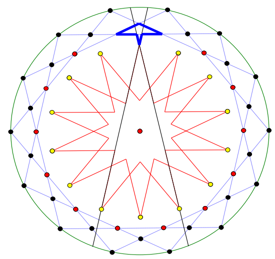

Figure 3b.

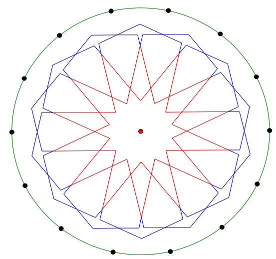

Figure 3c.

Finally, erasing the segments generated in the first step and erasing the innermost segments at the center of the design produces a star enclosed by arrow embellishments (**Figure 3c)**. This is Bourgoin's design that can be seen in his plates 164 - 167. For the remainder of this article Bourgoin's 14-star polygon along with the 14 arrows will be referred to as the "*14-star polygon*" or the "*star polygon*" or the "*star*."

Construction of Bourgoin's Designs

Plate 165.

Now that we have constructed the basic *14-star polygon*, we may use it to recreate Bourgoin's patterns. Although the plates in [1] are line drawings with no color, we have created a colored rendition of a group of four of the *stars* and the interstitial space using the *Geometer's Sketchpad* and *Paint* [8] software packages.

The pattern from Plate 165 is given in **Figure 4a**. Notice that the *stars* meet two at a time, with the arrow tip of one *star* just touching an arrow tip of an adjacent *star*. Also, each *star* is oriented so that two of the arrows are centered at the top, rather than as the *star* in **Figure 3c** where only one arrow is centered at the top. To achieve the design in the space between the *stars*, we extend the edges of arrows that border this space until they meet (see **Figure 4b**). All of the extended segments in this space are retained in order to finish the design. Once the requisite polygons between the *stars* are constructed, the rest of the design in Plate 165 can be completed by using symmetry and repetition.

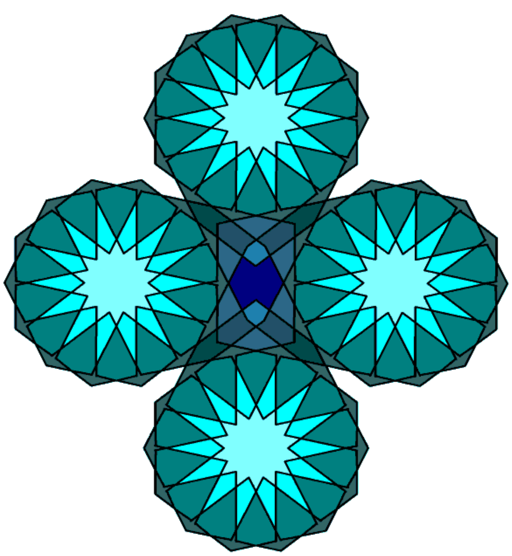

Figure 4a.

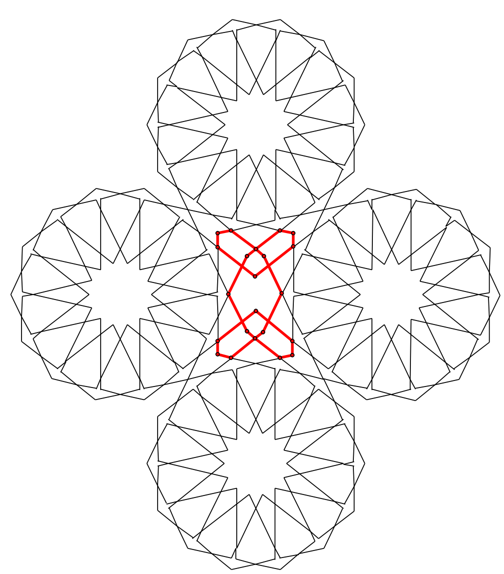

Figure 4b.

Plate 166.

Figure 5a shows a colored rendition of four of the *stars* and the tiled space between them in Plate 166. In this design, if the *stars* are surrounded by the circles used in our original construction in **Figure 3c**, each circle is tangent to two others, and the two middle circles overlap, enclosing their two arrow tips that touch (**Figure 5b**). Notice that all of the *stars* are oriented with two arrows centered at the top.

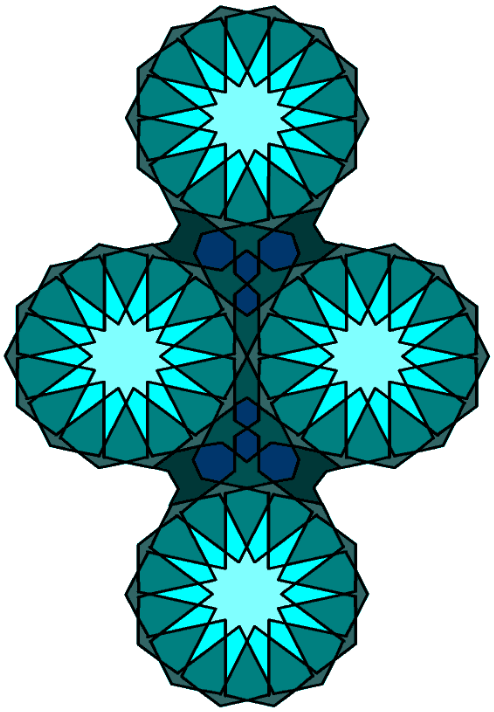

Figure 5a.

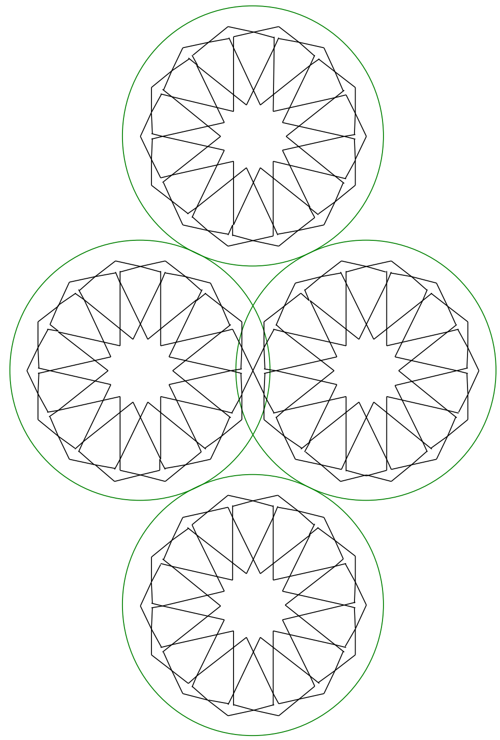

Figure 5b.

The design in the space between the *stars* contains four almost-regular heptagons and four smaller hexagons, as well as several nonconvex polygons to fill the remaining space. To construct these various polygons, we first extend the edges of arrows that surround the space (see **Figure 5c**). Intersections of these extensions determine some of the polygons and partially determine the hexagons and septagons. To complete the two hexagons, we find the midpoint of two vertical segments and then join their opposite vertices. Similarly, to find the vertices of the missing sides of the heptagons, we find midpoints of the dotted segments on which two sides lie (see **Figure 5d**). We can then complete the design joining the midpoints with line segments, and erasing the unneeded construction circles and line segments (see **Figure 5e**). Having found a way to construct the requisite polygons between the top three *stars*, the rest of the design in Plate 166 can be completed by using symmetry and repetition.

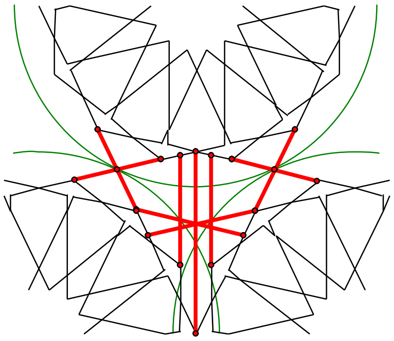

Figure 5c.

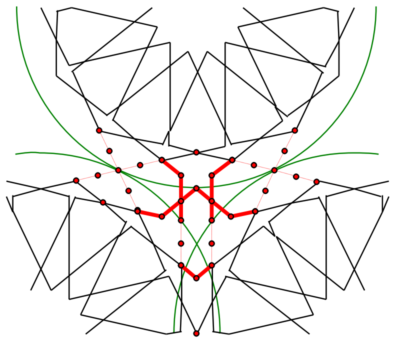

Figure 5d.

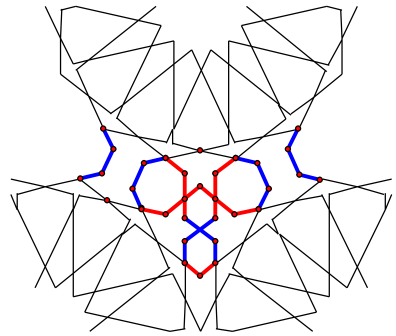

Figure 5e.

Plate 164.

Figure 6a shows a colored rendition of four of the *stars* and the tiled space between them in Plate 164. It was created in a manner similar to those we have already discussed. In this design, the arrows of adjacent *stars* overlap and merge to form pairs of 5-pointed stars. The lozenge-shaped petals of adjacent *stars* just touch. Notice that these *stars* are oriented with only one arrow centered at the top, which differs from the first two designs (**Figure 6b)**. The space enclosed by the four *stars* contains two almost-regular heptagons and some nonconvex polygons that surround these.

To produce the design in the space between the *stars*, we extend the edges of arrows that surround the space until they intersect other line segments comprising nearby arrows and then find midpoints of the dotted segments shown in **Figure 6c.** Finally, we create the three missing sides of the heptagons by joining existing points with line segments and then erase the unneeded line segments (**Figure 6d**). Symmetry and repetition can generate the rest of the design in Plate 164.

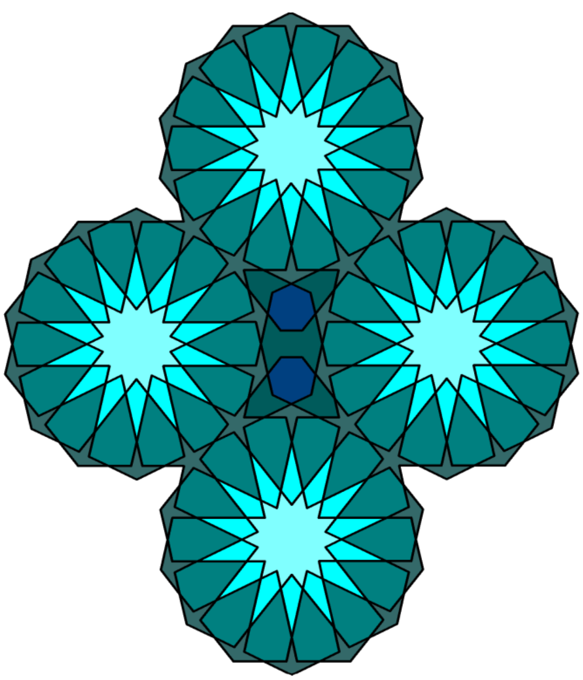

Figure 6a.

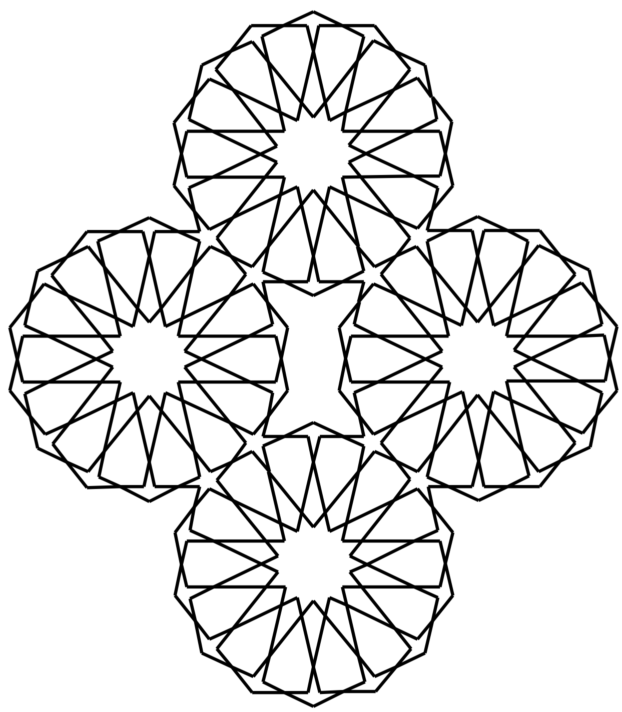

Figure 6b.

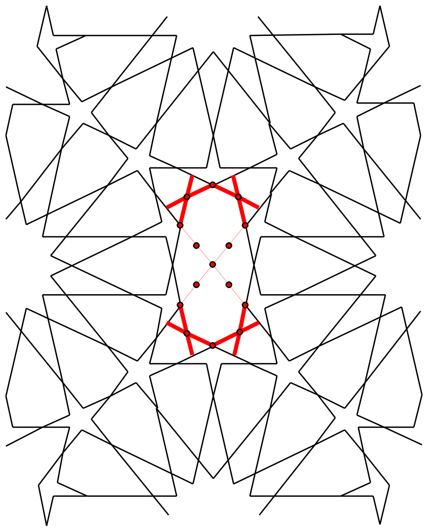

Figure 6c.

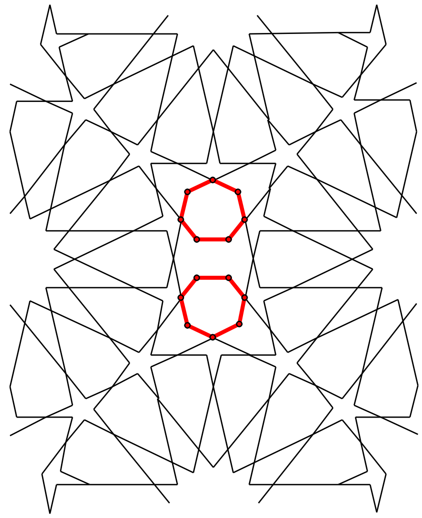

Figure 6d.

Plate 167.

Plate 167 of Bourgoin's *Arabic Geometrical Pattern and Design* is the last 14-pointed star polygon design to be considered. It is especially interesting since it also contains double 7-pointed stars as well as almost-regular heptagons. **Figure 7a** shows a colored rendition of a portion of the design. In this design, Bourgoin has separated the four stars so that none are touching. Between the *stars* there is a large area to be filled in with other polygons.

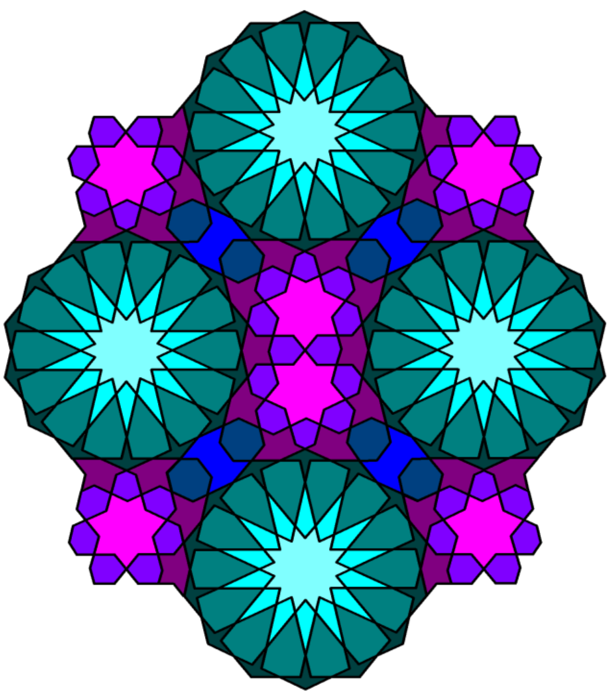

Figure 7a.

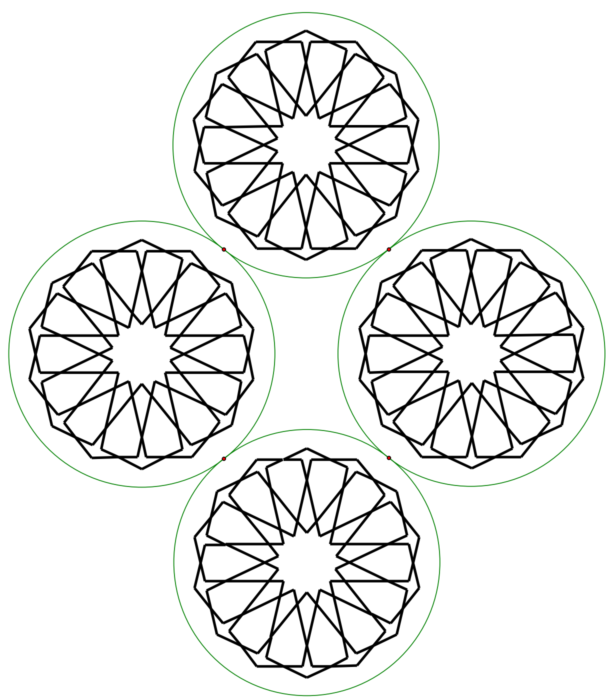

Figure 7b.

The design may be generated by first creating four of the *stars* each oriented with one arrow centered at the top and circumscribed within a circle, as in **Figure 3c.** The circles are positioned to be tangent in pairs, as shown in **Figure 7b**. To achieve the design in the space between the *stars*, we extend the edges of arrows until they meet other line segments (**Figure 7c)**, thereby creating the necessary points of intersection along the perimeter of this space. Some of these segments form the edges of the septagons, but most are no longer needed and so are erased. The midpoints of the remaining dotted segments, as shown in **Figure 7d,** are now found. Connecting the appropriate points produces the septagons in the upper half of the space (**Figure 7e**). Additional line segments drawn between existing points in **Figure 7f** yield a 7-pointed star in **Figure 7g.** Using symmetry and repetition, we can generate the rest of the design in Plate 167.

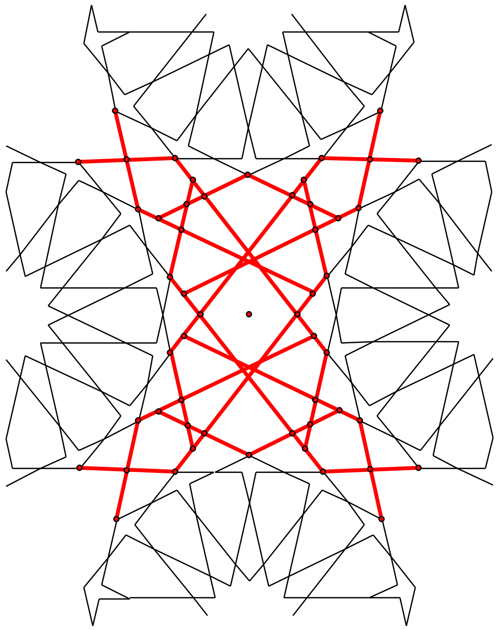

Figure 7c.

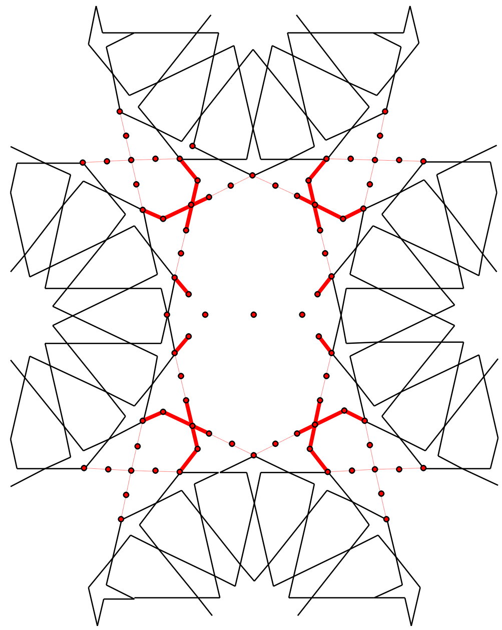

Figure 7d.

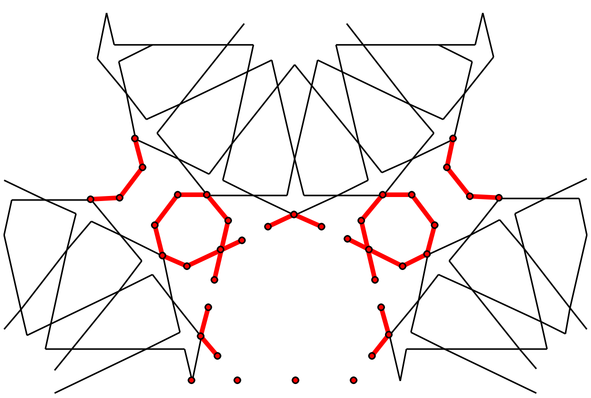

Figure 7e.

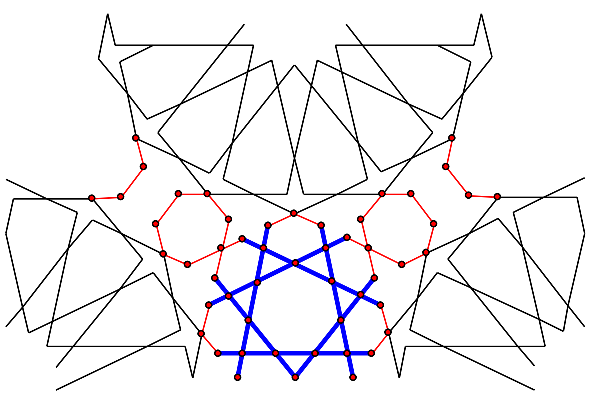

Figure 7f.

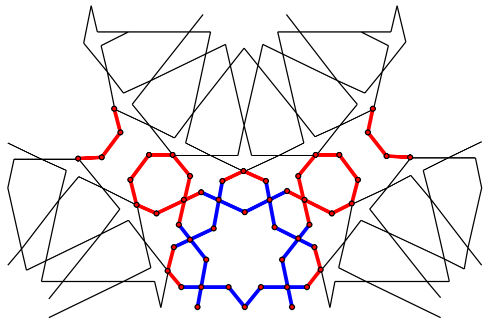

Figure 7g.

Observations

It is interesting to note that from the same {14/6} star pattern, Islamic artisans could create such different repeating patterns. In each case, four of these *stars* were clustered, and the space between them was filled differently. Bourgoin copied these designs from Islamic sources and lightly sketched some circles and lines that underlie the construction. But this is after the fact, an analysis of a completed design. He does not indicate how the design might have been achieved. What we have shown here is how, after determining the position of the four *stars* in a cluster for each design, the space between the *stars* might be generated using plausible compass-and-straightedge constructions. Although we thoroughly examined David Wade's collection of over 4000 images of Islamic patterns, available online [9], and the 114 Islamic architectural and ornamental design sketches of the *Topkapı Scroll* [10], hoping to find extant examples of these four patterns, we were unable to locate any.

Acknowledgements

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