Harmonious Dances

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Abstract

Although choreographers may employ mathematical principles when creating dances, overt attention is seldom given to the mathematics by either the audience or the choreographer. We will examine the mathematical elements in several of the author's recently choreographed dances in which mathematical ideas were embedded purposefully and with the intent that the mathematics be at least somewhat visible to the audience. Two of the dances included elements of proofs, and all but one are part of the show "Harmonious Equations," which premiered in 2008.

Introduction

Choreographers often use mathematical principles when creating dances, though usually little overt attention is given to these connections by either the audience or the choreographer. In this paper we will examine the mathematical elements in several of the author's recently choreographed dances. In these dances the mathematical ideas were embedded purposefully and with the intent that the mathematics be visible to the audience. Two of the dances included elements of proofs, and the all but one that we will look at are part of the show "Harmonious Equations," which premiered in 2008.

The Sum of the Internal Angles of a Polygon

In one case the dance utilized a kinesthetic "proof," perhaps better described as an explanation in movement, that the sum of the internal angles of an n-sided polygon is $\pi(n–2)$. The author and sarah-marie belcastro used the proof described below when they performed a short dance composition as part of a 2008 lecture demonstration [1]. The dance uses simple ballet or modern dance steps and turns.

Suppose the internal angles are $a_1,\ a_2,\ a_3,...,\ a_n$. Most proofs that the author has seen, such as those mentioned in Richeson [2], first sum the external angles $\pi - a_k$. If one "walks" successively along the edges of the polygon, then at the kth vertex one must turn through the external angle $\pi - a_k$, so the overall sum must be 2π, which is the result of walking all the way around the polygon:

$$(\pi - a_1) + (\pi - a_2) + (\pi - a_3) + ... + (\pi - a_n) = 2\pi$$

Then the algebraic rearrangement of this expression gives us

$$n\pi - (a_1 + a_2 + a_3 + \ldots + a_n) = 2\pi$$

or

$$a_1 + a_2 + a_3 + \ldots + a_n = n\pi - 2\pi = (n-2)\pi$$

Some of the algebraic steps in this proof might be replaced with kinesthetic "steps," as follows. Suppose one walks facing forward along the first side, from vertex v_7 to vertex v_1, turns through the internal angle of the polygon a_1, walks facing backward from v_1 to v_2, and continues in this way for one circuit around the polygon, alternately facing forwards and backwards along sides (Figure 1).

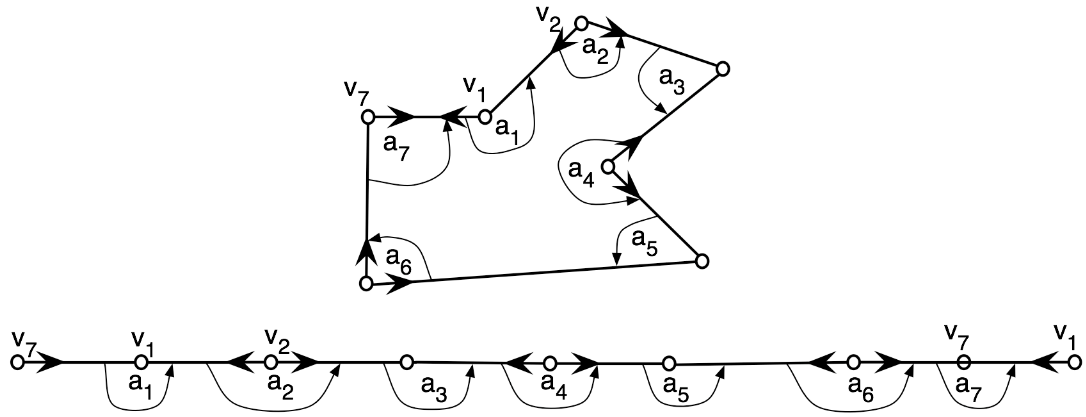

Figure 1: *Kinesthetic proof of the internal angle sum of a polygon.*

If there are an odd number of sides, as in the top diagram in Figure 1, then one ends up facing backwards on the first side after the last internal turn. Note also, in the example shown, the turns through the internal angles are all counterclockwise, while one circuit around the polygon proceeds clockwise. It is easy to see what the total sum will be if we imagine that the sides are all laid out along a straight line, as shown on the bottom of Figure 1. Then the total angle sum will be $n\pi$. However, in bending the path back to the shape of the polygon, so that the final copy of the first edge coincides with its first copy, and in the same direction, we will need to add one total turn in the opposite direction, which must have measurement -2π. Here we assume that the "winding number" in the clockwise direction is one turn, not multiple turns. So the total sum of the internal angles of the polygon must be $n\pi - 2\pi = (n-2)\pi$. This "proof" is easily demonstrated by dancing around the edges, and that is how we used it in the short 2008 composition, in which we proceeded around the edges of a triangle. Of course, here we have implicitly used the fact that the sum of the external angles, each measuring the difference between the straight line of angles and the internal angle that is its supplement, is 2π; but the experience of turning forwards and backwards n times, minus the ultimate 2π rotation, is accomplished physically.

Gauss-Bonnet formula. In that event in 2008, the author also danced a short circular dance phrase verifying that the sum of the angles of a spherical triangle obeys the Gauss-Bonnet formula. In this case the degenerate "triangle" had its vertices on the equater of an imagined unit sphere, so that each of the three turns at vertices measures π. Then the Gauss-Bonnet formula says that

$$\text{(triangle area)(Gaussian curvature)} + \pi = \text{angle sum}$$

Because the area of the sphere is 4π, this particular triangle with its vertices along the equater is really a hemisphere, and has the area of a hemisphere. Also, the curvature of the unit sphere is 1, so

$$(2\pi)(1) + \pi = \text{angle sum} = 3\pi$$

It would be interesting to find a kinesthetic demonstration of the Gauss-Bonnet formula that works for a variety of surfaces, not just a demonstration of one instance.

Harmonious Equations

A series of short mathematical dances were created by the author as part of "Harmonious Equations," directed by Keith Devlin [3], in December of 2008. In this hour-length show Keith gives short verbal explanations of seven important equations, which are then translated into songs composed and sung by the a cappella choral group Zambra. A trio of dancers performs to the musical composition. In the remainder of the paper we will look at the mathematics embedded in these dances.

Pythagoras' Theorem: $a^2 + b^2 = c^2$. In the duet dance accompanying this equation our dancers used the dissection proof of the Pythagorean theorem found by Henry Perigal (1801-1898), shown in Figure 2 (Frederickson [4]). It is based on a dual tiling of the plane, first by the smaller two squares of sides a and b, then overlaid by a tiling by squares with sides equal to hypotenuse c. In the dance we used squares of sides 3, 4, and 5 to connect with Zambra's use of this classic right triangle within their song. This dissection proof is also very similar to that of the Arab mathematician Thabit Ibn Qurra (836-901 AD).

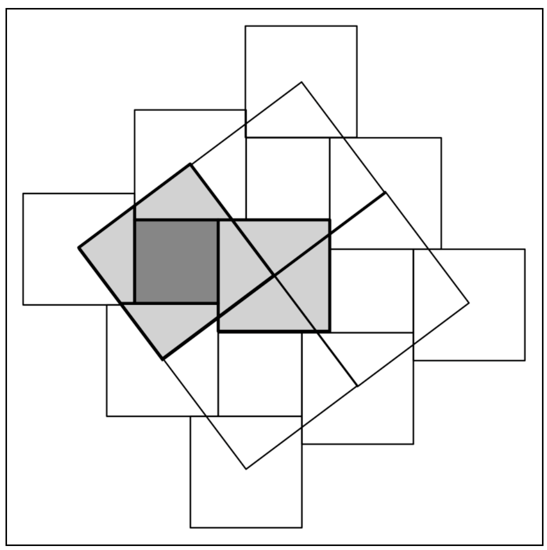

Figure 2: *Perigal's dissection proof of the Pythagorean theorem.*

In this duet, each dancer wields two Styrofoam quadrilaterals, shown light gray in Figure 2; the smaller square with side 3, shown darker here, is mounted on a post, on which it rotates. The dancers make a variety of shapes with the quadrilaterals, finally making the side 4 square, then surrounding the side 3 square to make the side 5 square. In future performances we plan to find a way to make visible to the audience the fact that we are working with the 3, 4, 5 right triangle.

Area of a Circle: Area = πr^2. We can see how much longer the circumference of a circle is than its diameter, by forming our arms into a circle, then spreading them wide. This also gives us an approximate value for π. The joints of the shoulder and arms are such that one may swing the arm in a circle, with the arm functioning as the radius around the central point at the shoulder joint, and it is fun to play with ways the two arms might make such circles in the air at the same time. For this section of Harmonious

Equations we utilized synchronized arm circles, static circles with the arms, and circular floor patterns to create a composition suggesting various danced aspects of circles.

Einstein's equation: $E = mc^2$. We certainly could not use radioactive decay, the phenomenon explored in Einstein's 1905 paper that contained the famous equation, so we chose the next best thing, fluorescence, the understanding of which was greatly furthered by another of Einstein's 1905 papers on the particulate nature of light. In this section we used lines "painted" with white cloth tape on black costumes, and illuminated by black lights, while creating shapes that shimmer and finally "explode," in accompaniment to Zambra's score.

Leibniz's series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$. This series is also known as the Madhava- Liebniz series, after the mathematician Madhava Sangamagrama of Kerala, India (1350-1425), whose discovery preceded that by Gottfried Liebniz (1646-1716) by 300 years [5]. We played with the swing time of Zambra's music for this section by using a rhythm game popular among children. In that game, taught to the author by 13 and 14 year old girls when he was dance faculty at a summer ballet workshop, the participants repeatedly play a short rhythm on plastic water bottles, which they pass one person to the right around a circle at the end of each rhythmic phrase. In our dance we used 8 inch "gator foam" (hardened foam core) cubes, showing four sides to the audience. One side is colored yellow, one blue, one black, and the fourth is also black but includes a portion of the Liebniz equation.

We developed a complicated set of switches of the boxes, performed while playing rhythms based on those of the water bottle game. Keeping track of these switches involves symmetry group theory, an area of mathematics concerned with such "transpositions." Though perhaps not directly relevant to this equation, group theory was developed by the mathematician Galois to decide whether certain families of equations are solvable at all.

The boxes are placed in a line on a table facing the audience, and manipulated by three dancers, each of whom grabs a pair of boxes during each of ten short sections of the dance. The dancers slap out rhythms on the boxes while switching their order and also turning one or both 90 degrees. The sequence of ten location switches, for the moment disregarding the turning of the boxes, is shown below using cycle notation. The boxes are numbered 1 through 6 in their original positions, and "*" is used to show composition of permutations, which really means that one permutation immediately follows another. In cycle notation, for example, (135642) indicates that the box in position 1 moves to position 3, 3 moves to 5, and so on, until finally 2 moves to 1. The sequence used in the dance is:

(12)(34)(56) * (12)(34)(56) * (12)(34)(56) * (12)(34)(56)
* (135642) * (identity) * (identity) * (12)(34)(56) * (12)(34)(56) * (12)(34)(56)

Because (12)(34)(56) * (12)(34)(56) = identity, the result of these ten permutations is really just (135642) * (12)(34)(56) = (14)(36) = (14)(36)(2)(5), where (2)(5) is included to indicate that boxes 2 and 5 really end at their starting positions. The final order of the boxes is thus 426153.

The process is further complicated by the fact that on each move each dancer might easily turn one or both boxes 90 degrees clockwise or counterclockwise, producing a variety of colored patterns in the line of boxes. If we let B = black facing, Y = yellow, L = blue, E = equation, the sequence of turns created the following patterns, in order:

LLLLLL,YYYYYY,LYLYLY,LLLLLL,LLLLLL,BLBLBL,

BBBBBB,BEBEBE,BBBBBB,BEBEBE,EEEEEE

The author used this symbolic process to determine the order to place the boxes in at the start of the dance, though the other dancers solved the problem more easily by simply running the dance once and adjusting the starting position according to the locations of the boxes at the end!

Euler's polyhedron formula: V–E+F=2. In this section the dancers use the "hexastar," a hexagon with an extra free edge attached at each vertex, shown in Figure 3, to create a cube, octahedron, two linked tetrahedra, and other shapes (Figure 4). The hexastar is made of 1-inch diameter foam-covered PVC pipe, in 40 inch sections, joined at the vertices by bungee cord (see [6] for further discussion of this prop.) We were looking for one set of twelve PVC pipes that could easily be wielded by 3 dancers. Can the reader visualize how to fold this structure at its vertices in order to do this? (Solutions on last page, Figure 6.)

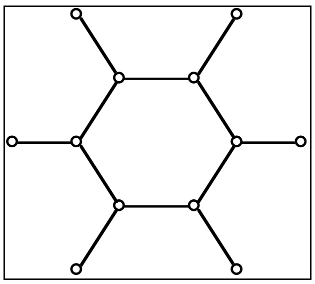

Figure 3: *Hexastar.*

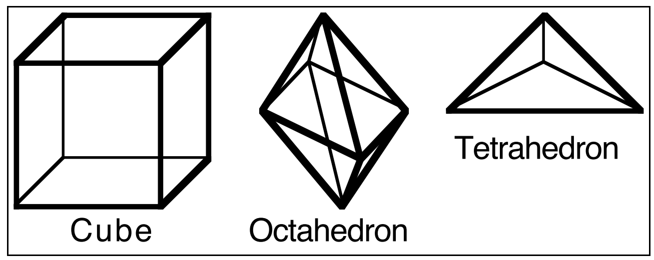

Figure 4: *Platonic solids formed with the hexastar.*

The 6 faces of the cube and 6 vertices of the octahedron are combinatorially related to the 6 edges of the tetrahedron, as suggested in Figure 5. These connections are also used by Zambra in their song,.

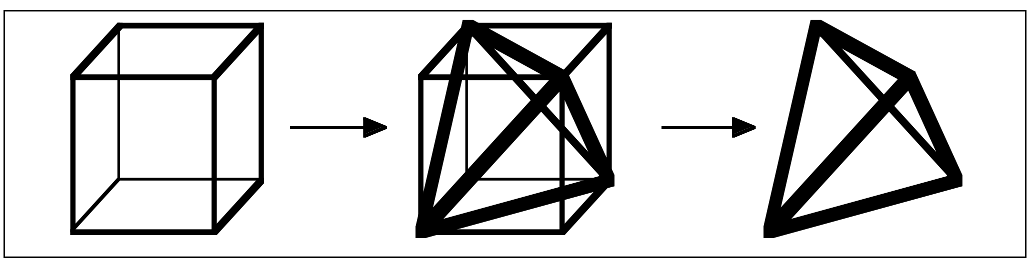

Figure 5: *The cube and the tetrahedron.*

References

[1] sarah-marie belcastro and Karl Schaffer, "Dancing Mathematics and the Mathematics of Dance," AMS/MAA Joint Meetings, San Diego, January, 2008.

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[3] Keith Devlin, Zambra, and Karl Schaffer, "Harmonious Equations," Dec., 2008, Asilomar, CA.

[4] Greg Frederickson, *Dissections Plane and Fancy.* Cambridge Univ. Press, 1997.

[5] J. J. O'Connor and E F Robertson. "Madhava of Sangamagrama". School of Mathematics and Statistics University of St Andrews, Scotland. http://www-gap.dcs.st-and.ac.uk/~history/Biographies/Madhava.html. Retrieved on 2009-02-02.

[6] Karl Schaffer, "One-dimensional Origami: Polyhdral Skeletons in Dance," in Origami^4, ed. by Robert Lang, A.K. Peters Press, expected publ. date 2009.

How to fold the hexastar into the cube, octahedron, and tetrahedra. The cube and the octahdron each have 12 edges, as does the hexastar, so no edge duplication is necessary. Also, since the tetrahedron has 6 edges, two tetrahedra will have 12 edges, and no edges are duplicated. It is possible to fold it into one tetrahedron with each edge doubled as shown at the bottom of Figure 6.

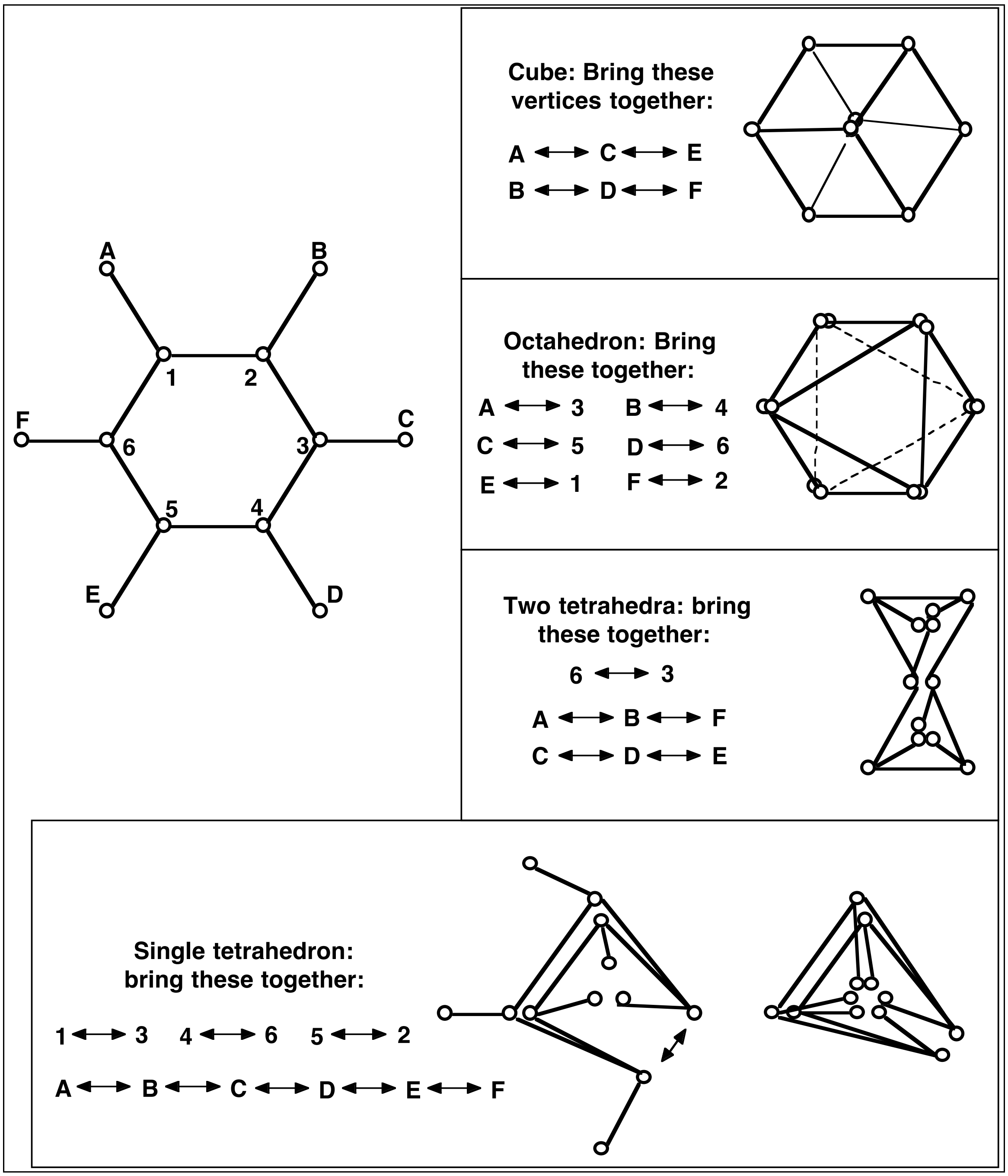

Figure 6: *How to fold the hexastar into cube, octahedron, and tetrahedra.*