The Frustrated Mathematician: A Call to Artists

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Abstract

One source of frustration for some mathematicians is their inability to express sophisticated and beautiful ideas in a language more accessible to the broader community. Five examples of mathematical concepts are provided which lend themselves to artistic interpretation and inspiration, in areas ranging from precalculus to real analysis and differential equations, along with capsule descriptions of the mathematical processes and thumbnail graphics to help illustrate ideas difficult to capture in equations alone. The hope is that these ideas could serve to inspire artists and musicians who are more capable than most mathematicians in alternative forms of expression.

Five Favorite Frustrations

Introduction. Over the past eight years, an artist colleague (Keith Lord) has team-taught an advanced interdisciplinary general education course with me, entitled "The Mysterious Dance of Art, Mathematics, and Music", which we collaborated on developing. We have eight times now experienced anew the joy, wonder, and fascination of learning about the content, methods, and ways of knowing in our disciplines. As wonderful as this experience has been, one source of frustration has been my attendance at Bridges conferences where I have realized that my utter and complete lack of competence in all things artistic prevents me from realizing in any artistic medium ideas inspired by mathematical concepts or problems encountered in the course of earning my livelihood as a mathematician. One possible solution to quelling this frustration might be to share some of those ideas with artists who have an interest in communicating and giving expressive form to mathematical ideas, and what better way to accomplish that than by sharing these thoughts with the mathematically inclined artists and musicians at a Bridges conference? Very few artists and musicians have formal advanced training in mathematics (think Helaman Ferguson as the canonical counterexample), but there is a large set of artists living in the complement of my space, those people who are thoroughly versed in some mode of artistic expression but who search for nuggets of mathematical inspiration and understanding to push their creativity toward new boundaries. It is to these folks that I address this paper.

To Infinity and Beyond. I first encountered the system of nonlinear ordinary differential equations $\frac{dx}{dt} = y - x\sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}}$, $\frac{dy}{dt} = -x - y\sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}}$, as a graduate student over three decades ago, but I still recall the wonder with which I regarded the solution curves and limit cycles as described on the chalkboard by my professor (since it was not at all an easy task back then to actually produce a high-resolution graphic plot of such a solution). The solution consists of an infinite number of limit cycle circles of radius $1/(n\pi)$, where the space between any two limit cycles is completely filled by an infinite number of non-intersecting spirals which both spiral inward toward the next smallest limit cycle and outward toward the next largest limit cycle (see Fig. 1). This is one of the most amazing visual expressions of infinity that I have ever encountered, and certainly deserves a more creative form of expression than simple output from a computer plotter.

Bridge to Infinity. A standard problem in many calculus books illustrates the extremely slow yet inexorable growth of the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ toward infinity. The problem is to stack as many blocks on top of each other so as to produce the largest overhang possible without the stack of blocks tumbling down. The maximum overhang is achieved when for n stacked blocks, the kth block extends past the block below it by $\frac{1}{2k}$ blocks for a total overhang of $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2(n-1)}$ blocks, which of course approaches infinity in the limit, thus permitting extensions of any length desired (see Fig. 2). This appears to be a prime candidate for some type of creative sculpture or painting.

Lunar Origami. This example was born in a general education mathematics course, when to illustrate the rapid growth of the exponential function $f(x)=2^x$, I offered a \$20 bill to any student who could fold a sheet of notebook paper in half, then in half again perpendicular to the original fold, and so on, for a total of eight folds. To illustrate just how quickly that function grows, on the spur of the moment I gave them a problem for extra credit to determine the exact number of folds in a piece of notebook paper, which if they could be carried out, would produce a thickness of paper which would reach from the earth to the moon. Assuming a ream of 500 sheets of paper has thickness two inches, the answer is just 42 folds, and a "Folding to the Moon" painting or sculpture conceptualizing these 42 folds in some way seems natural.

Noncommutative Sums. The lecture at my very first job interview provided a demonstration that the equality $\sum_{m=1}^{M}\sum_{n=1}^{N} f(m,n)=\sum_{n=1}^{N}\sum_{m=1}^{M} f(m,n)$ does not necessarily hold when $M=N=\infty$, a rather counterintuitive and surprising result. Divide the surface area of a cylinder into m bands of equal width, with each band broken down into n congruent isosceles triangles, then sum the area of the triangles and let the number of bands (m) and the number of triangles per band (n) approach infinity in different orders. The result will be two very different answers for the surface area (see Fig. 3).

Fubini Vindicated. The final example illustrates that exchanging the order of integration in a double integral may resulting in different answers if conditions necessary for applying Fubini's Theorem are not satisfied. If we define a function $f(x,y)=a_{ij}$ on the rectangle $b_{ij}=\left(\frac{1}{i+1},\frac{1}{i}\right)\times\left(\frac{1}{j+1},\frac{1}{j}\right)$ where $a_{ij}=\frac{1}{2^{i-j}}$ if $i>j$, $a_{ij}=-1$ if $i=j$, and $a_{ij}=0$ if $i<j$, then interchanging the order of integration yields $\int_0^1\int_0^1 f(x,y)\,dxdy=0\neq-2=\int_0^1\int_0^1 f(x,y)\,dydx$. (See Fig. 4 for a rough illustration of the function $f(x,y)$.) The notion that summing the heights of columns in two different directions produces different results is an idea begging for an artist to approach it with more finely tuned visual aesthetics.

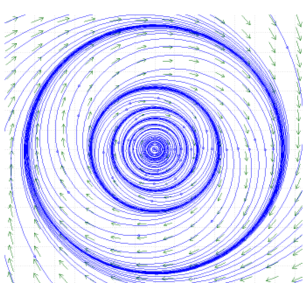

Figure 1: *Circles and spirals*

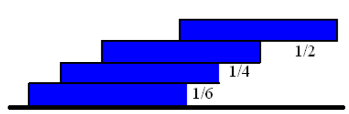

Figure 2: Bridge to infinity

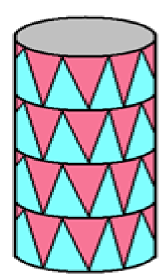

Figure 3: S*ums*

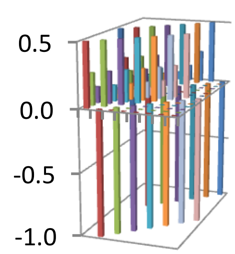

Figure 4: *Fubini*

Conclusion. It is the hope of this author that these examples will serve to pollinate the minds of artists and musicians with mathematical notions which in some way may spawn creative reflections on significant mathematical concepts, and ultimately lead to artistic works which might otherwise not receive the benefit of the perspectives and expressionist forms of painters, sculptors, weavers, musicians, and other artists.