Mathematical Experiments with African Sona Designs

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Abstract

The sona designs of the Chokwe people of Angola/Congo are a particularly attractive form of "mirror curves," which can be visualized as a curve drawn through a lattice of dots, bouncing off the edges of the lattice and off "mirrors" placed between some of those dots. Most such designs are drawn as a single, uninterrupted line, and most contain symmetries of some type. The mathematics of the designs reflect issues of common divisors of numbers, Eulerian cycles, and symmetry groups. This talk will investigate ways to use a sona-drawing program to investigate these topics with students from middle school through college, with particular emphasis on the experimental discovery of mathematical facts. The sona program developed by the author is cross-platform and free.

Introduction

The Chokwe people of Angola and Congo have a drawing tradition, done both in sand drawings and on more permanent objects, that has attracted a substantial amount of mathematical attention (e.g. Ascher [1], Gerdes [4, 5], Jablan [6]). Although their sona (singular "lusona") drawings arise in several different forms, one of the more common, and the most mathematical, can be viewed as a grid of dots with a curve passing through the grid, "bouncing" off the boundary and off internal "mirrors," to create a single continuous curve. The "Leopard with Cubs" lusona shown in figure 1 is an example of such a drawing without internal mirrors. This sand drawing includes a few features added to the fundamental design to indicate the heads and tails of the mother leopard (vertical) and her two cubs (horizontal). Figure 1 shows it in three forms: a photograph of the drawing made in the sand, a computer rendition of the drawing showing it with the heads and tails of the leopards, and a computer drawn form of the underlying Eulerian drawing, as the sona drawing program draws it. This computer version does not include the artistic additions used to personify the animal forms the artist has seen in the principal monolineal drawing.

The "Leopard with Cubs" is a sona example with no internal "mirror" walls for the curve to bounce off. A classic example of a sona that can be modeled by using such mirrors is the "Chased Chicken" of figure 2, so called because it seems to capture the path that a chicken might take if you were trying to catch it. Figure 2 shows both the sona as it would be drawn by a Chokwe artist, and the "hidden" walls that help to define the shape. The artist making this drawing lays down the grid of dots shown here, does *not* draw the walls, but rather draws the Chased Chicken sona directly on the dot grid. There is some evidence that the Chokwe artist envisions the walls in the dot layout as they are drawn in the program, in the same way that mathematicians investigating such sona have studied such wall placements. In many cases, the wall layouts themselves help us to see more clearly the symmetries within the sona design. Both parts of figure 2 were drawn by the sona program.

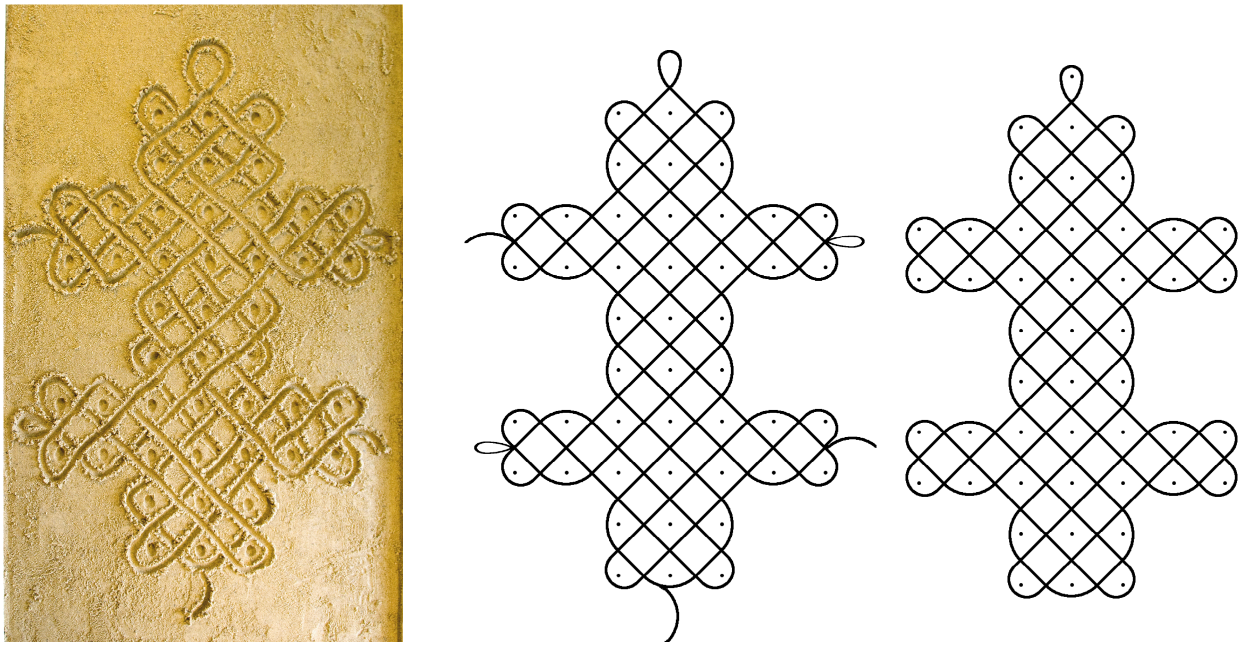


Figure 1: *The "Leopard with Cubs" lusona, as it would be drawn in the sand, with additional heads and tails of the mother leopard (horizontal) and her two cubs (vertical). On the right is the underlying monolineal curve, as drawn by the sona program.*

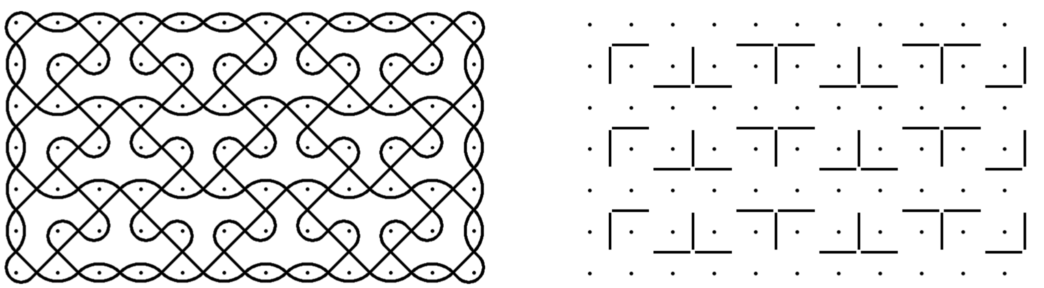


Figure 2: *The "Chased Chicken" lusona, on a 7 X 12 rectangular grid, along with the mirror walls that define the curve. The design has pmg wallpaper symmetry.*

The range of possible sona designs gives rise to many different types of investigations, both mathematical and artistic. The use of a computer program to try different layouts of both dots and walls allows a student to imitate what might be months worth of work by a Chokwe artist while looking for interesting designs. Of course the fundamental problem is that not all sona grids will give rise to acceptable sona drawings: they may not be monolineal, or to make them monolineal may seem to require modifications (such as additional walls) that disrupt the symmetry or overall aesthetic appearance of the design. Our experience, with students from 4th grade through college, is that these experiments and open-ended design questions are enjoyed greatly by the students, many of whom continue to work on designs well after the workshop or homework assignment has been completed. (For many students, the ability to complement this computer work by trying a few of their designs on a sand table adds substantially to the experience.)

Mathematical Investigations

Greatest Common Divisor Investigations. The most basic experiment with sona is to experimentally determine for what dimensions an m X n rectangle, with no internal walls, will give a monolinear sona. The correct answer is "when m and n have no common factor," i.e. when gcd(m, n) = 1 (where gcd is the greatest common divisor). In most cases, the students' original hypothesis is that it happens whenever one of m and n are odd. A little experimentation, occasionally with suggestions of rectangle sizes to try, generally leads the students to the correct answer. (For young students, this is especially true if they have recently been simplifying fractions.) A similar investigation is to ask "If a rectangle does not give a monolineal sona, can you tell from the dimensions how many lines it takes to draw the sona?" (Answer: It takes gcd(m, n) lines.)

Some of the African sona designs are based off combinations of rectangles, such as *abutting rectangles* or *overlapping rectangles*, such as shown in figure 3. This leads to easy extensions of the basic rectangle investigation just mentioned. For example, we can ask "When do rectangles abutting (i) along a single dot, or (ii) abutting along two dots, give monolineal sona?" (Answer: (i) If both rectangles have dimensions with gcd's of 1; (ii) If one rectangle has a gcd of 1 and the other has a gcd of 2.) Similar, but slightly more complicated answers, apply to rectangles overlapping by 1 or 2 dots.

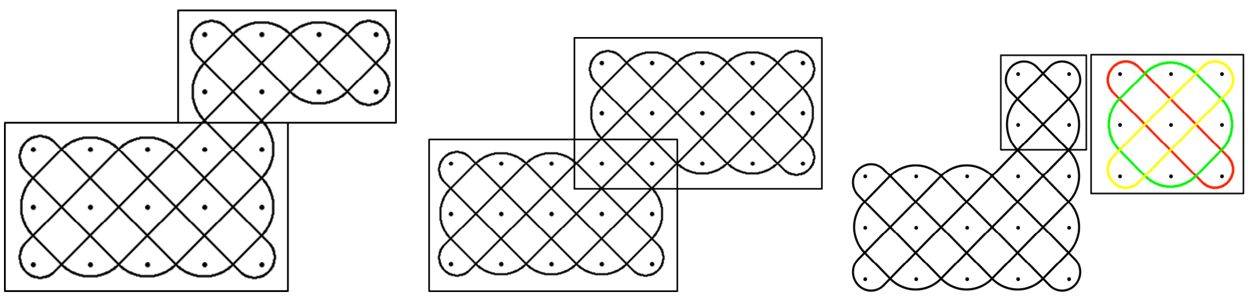


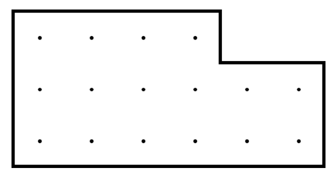
Figure 3: *Monolineal sona built from abutting and overlapping rectangles. On the left, two rectangles "abut at two dots," while the ones in the center "overlap at two dots." On the left is a re-interpretation of the center sona built by attaching squares to one of the rectangles.*

Attaching Rectangles. Another basic investigation is to ask what kinds of rectangles can be added to a monolineal sona, along one side, so that the result is still monolineal. Students quickly realize that they can add any 1 X n rectangle so, for example, the left sona in figure 3 is easily converted into an animal with legs and a tail, while still being monolineal. (Actual Chokwe sona often are modified to represent animals, and younger students especially seem to enjoy creating such animal representations.) With a little prodding, students can often discover for themselves that adding any square of dots along the side of a monolineal sona will result in a monolineal sona. For example, the left sona of figure 3 can be viewed as a 3 X 5 monlineal sona, with two 2 X 2 squares added in succession. The Chokwe artists themselves seem to have realized this fact, at least with 2 X 2 sona, as can be seen from the Leopard with Cubs sona earlier. The Leopard with Cubs can be constructed from a 3 X 10 rectangle (hence a monolineal sona) with successive 2 X 2 squares added, each of which leaves the original sona monolineal.

Once we have discovered that squares can be added along one side without changing monolineality, we can apply this process in reverse to see how many lines it takes to draw a sona on a particular grid. For example, with the dot layout in the center of figure 3, we can imagine amputating (cutting off) the 3 X 3 square from the upper right (without changing the number of lines), as shown on the right in figure 3, then amputating the 2 X 2 square boxed in, leaving ourselves with the single 3 X 5 rectangle, which we know to be monolineal. If we apply this amputation technique to a rectangle with sides m and n, this process can be seen to be a geometric analog of the Euclidean algorithm to find the common divisor of m and n. For example, with a 6 X 21 rectangle, we would amputate a 6 X 6 square three times to leave a 6 X 3 rectangle, then amputate a 3 X 3 square from that, leaving a 3 X 3 square (which is easily seen to require 3 lines to draw) and hence we have geometrically shown that the GCD of 6 and 21 is 3.

Additional GCD Investigations: The following exercises all have answers that can be discovered through experimentation, and whose answers are simply enough stated to be found empirically. As with the examples above, these investigations are usually too tedious to do by hand, but fairly easy to do with computer assistance from the sona drawing program. In general, they also have proofs which can be found by a mathematician, but the experimental discovery of these facts is accessible to all students.

For several of these questions, we refer to a "bite" being taken out of the corner of a rectangle. By this we mean removing, or erasing, some dots from an initial rectangle. The dot layout on the right, for example, would be viewed as a 3 X 5 rectangle with a 1 X 2 "bite" taken out of the corner. Of course there are two different ways to take that bite, horizontally or vertically. Generally, though, there is no difference in the answers that depends on that choice.



1) If a single dot is erased from a corner of a rectangle, i.e. we remove a 1 X 1 bite from the corner, when will the resulting sona be monolineal? (Answer: If, and only if, the rectangle dimensions have a gcd of 2.)
2) If a rectangular grid has dimensions with a gcd of r, $r > 1$, can you remove a 1 X k bite from the corner to create a layout with a monolineal sona? If so, what is the smallest such "bite" you need to take out to make it monolineal? (Answers: Yes. When $k = (r–1)$.)
3) If a rectangular grid has dimensions with a gcd of r, $r > 1$, can you take out a k X k bite from the corner to create a layout with a monolineal sona? If so, what is the smallest such "bite" you need to take out to make it monolineal? (Answers: Yes. When $k = (r–1)$.)
4) If a single dot is erased from an edge of a rectangle, but *not* at a corner, when will the resulting sona be monolineal? (Answer: If the rectangle dimensions have a gcd of 3, and the dot is the k-th one from a corner, where (k mod 3) = 2. It can also happen with *some* of the dots when the gcd is 1, but we don't know of a way to decide ahead of time which dots will work then. No other gcd's work.)
5) If two rectangular grids with gcd's of r and s respectively are aligned so as to overlap at one corner dot, can you tell how many lines the resulting sona will need? (Answer: $r + s – 1$.)
6) If two rectangular grids with gcd's of r and s respectively are aligned so as to overlap at two corner dots, can you tell how many lines the resulting sona will need? (Answer: If $r + s \geq 4$, it's $r + s – 3$; if $r + s = 3$, it's 2; and if $r + s = 1$ it is either 1 or 3, but we know of no criteria to decide when it's 1 and when it's 3.)
7) If you have two identical rectangular grids, with a gcd of r, can you overlap them in a 1 X n rectangle so the resulting sona is monolineal? (Yes; they should overlap in a 1 X r rectangle.)
8) If two rectangular grids with gcd's of r and s respectively are aligned so as to touch at two dots, how many lines will the resulting sona will need? (Answer: If $r + s \geq 3$, it's $r + s – 2$; otherwise, it's 2.)
9) If you have two rectangular grids with gcd's of r and s, can you align them to touch at k dots to make a monolineal sona? If so, what is the smallest value for k that works? (Answers: Yes. $k = r + s – 1$.)
10) By starting with a monolineal rectangle sona, and then attaching squares of dots, what interesting sona (e.g. animal or naturalistic shapes) can you construct? Look at examples of authentic Chokwe sona for ideas of shapes that might be appreciated by those artists.

Mirror Investigations. A basic result of adding mirrors (see Chavey & Straffin [3]) is that if we have a sona design with one or more lines, then adding a mirror at a place where two different lines cross will merge those lines into one (resulting in one fewer total lines), while adding a mirror at a place where one line crosses itself will split that line into two (resulting in one more total lines). The sona drawing program draws multiple lines in separate colors, so it's easy to locate places where adding mirrors will reduce the number of lines. In particular, we can use this idea to construct a monolineal sona while meeting the Chokwe aesthetic of symmetry. For example, in figure 4 there are many ways to find symmetric placements of two walls so that each of those walls merge two of the three colors, resulting in the desired monolineal sona. One such placement is shown on the right. With these 2 walls, the left wall would merge the pink and yellow lines into one, then the second wall would merge the black line with the pink-yellow line, resulting in a single line sona. In open-ended investigations, good students often construct a tentative sona, then use this technique to change the lusona to make it monolineal.

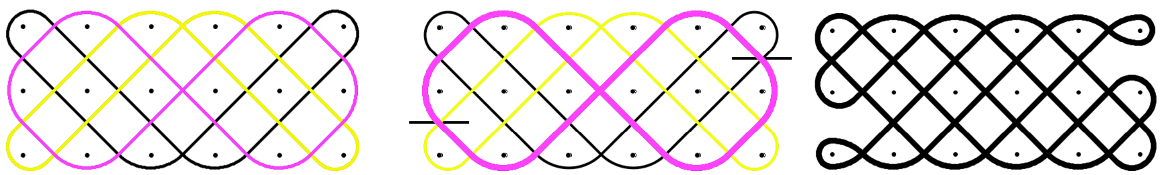
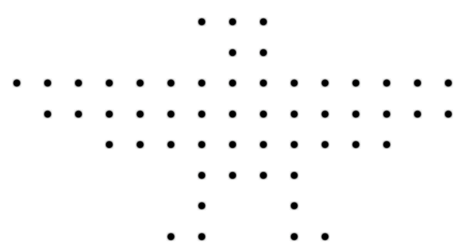


Figure 4: *The tri-lineal sona constructed on a 3 X 6 rectangle grid, on the left. This sona can be made monolineal by many different choices of 2 symmetrically placed mirror walls. One placement, showing the lines which are merged together, and the resulting sona, are shown.*

Sample experiments for this type of investigation might be:

11) Construct a sona layout that appears to model an animal or natural shape (e.g. from experiment #10). If the layout does not create a monolineal sona, find places to add walls so as to merge pairs of lines into a monolineal sona, while maintaining the naturalistic shape you designed.

12) Starting from a symmetric shape which does *not* create a monolineal sona, find places to add walls so as to merge all lines into one while still maintaining the symmetry of the original layout. One example might be the "eagle sona" on the right, which has bilateral symmetry except for the beak. This generates a 4-line drawing, which can be converted into a monolineal drawing by the addition of 3 walls while maintaining the bilateral symmetry.



Symmetry Group Investigations. To imitate the aesthetic of the Chokwe artists, we want to construct symmetric sona. Many of the authentic Chokwe sona have bilateral or rotational symmetry, although often with disruptions to the symmetry created by adding details to a symmetric sona to make the sona picture represent an animal or other natural object (such as horns, beaks, tails, etc.). Creating such symmetric sona is an enjoyable investigation by itself. One particular area for interesting, open-ended artistic exploration is the construction of sona with wallpaper pattern symmetries. The Chased Chicken design shown in figure 2 is one such example. Figures 5 and 6 show other examples from the Chokwe. Several other examples of invented wallpaper-pattern sona can be found in Chavey [2].

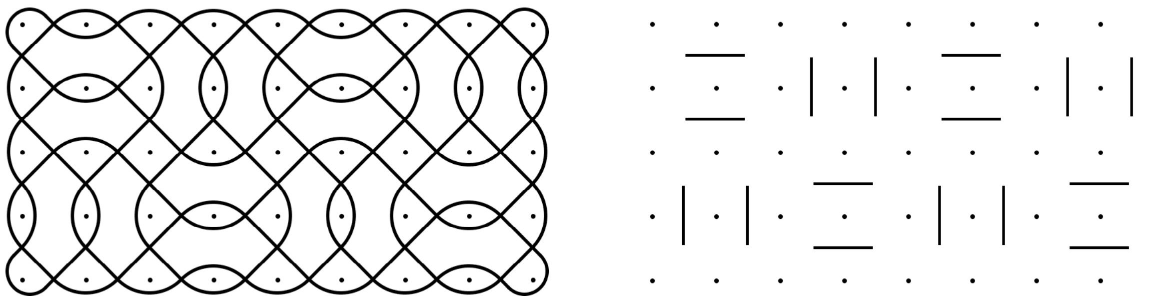


Figure 5: *A monolineal sona with p4g symmetry, showing the sona (left) and the walls defining it (right). This sona design will be monolineal for all dimensions* $(4m+1) \times (4n+1)$.

The Chokwe sona of this type share a couple of important characteristics:

a) They have a one-dot "frame" around the design, i.e. every dot on the outside edges of the design is connected, by the sona, to the dots on either side along that edge, i.e. there is no "wall" that cuts across this frame;
b) No side of the frame has a two-dot "block" along the edge, i.e. along every side, some of the walls reach to within one row or column of the edge of the design;
c) Excluding some boundary effects along this one-dot frame, the internal portion of the design has a "wallpaper" symmetry group, i.e. one of the 12 symmetry groups of the plane that does not include 3-fold or 6-fold rotations.

These examples give rise to several areas of investigation:

13) The design of figure 5 uses a pair of walls (| |) as a building block to construct a 2-dimensionally symmetric pattern, while the Chased Chicken design uses a T shape (truncated on the sides) to construct such designs. What types of simple building blocks can we use to construct such designs? (Some students have found examples answering this in 40–60 minutes, but I think of this, and the other questions in this group, as part of a longer exploration. I generally have students work in pairs.)

14) For a given building block (a.k.a. "fundamental region"), what wallpaper symmetry groups can be applied to that design to give monolineal sona? How far apart should those blocks be spaced, i.e. how long should the translations be with respect to the block of walls?

15) If we succeed in constructing a monolineal sona by repeating a particular wall configuration, can we tell, empirically, what other sizes of rectangle will generate such designs? For example, the pattern of figure 5 works for any rectangle of dimension $(4m+1) \times (4n+1)$, for any value of m or n. The "Lion's Stomach" design of figure 6 works for a rectangle of width $4n+1$ for any n and any height.

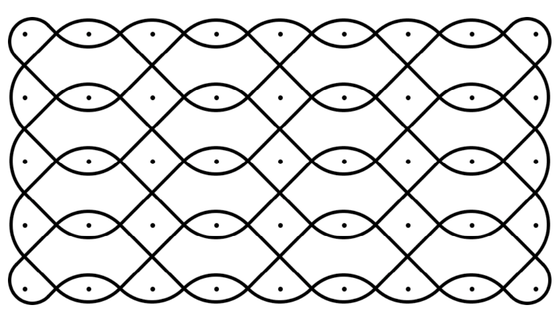
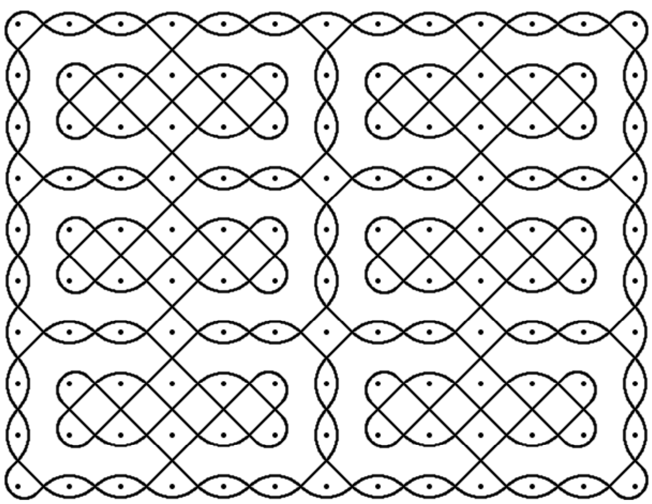


Figure 6: *The sona called "Lion's Stomach" by the Chokwe. This has pmm symmetry and is monolineal for all dimensions* $m \times (4n+1)$.

Not too surprisingly, some of the results of this type are more challenging to discover than others. For example, to construct a Chased Chicken layout requires a rectangle of dimensions $(2m+1) \times 4n$. But this will be monolineal if and only if $\gcd(m+1, 2n+1) = 1$ (see Chavey [2]). This result is noticeable more difficult to discover empirically than results such as those for figures 5 and 6.

In a similar vein, some fundamental regions of walls are easier to analyze than others. We have found, for example, (see [2]) that the wall pattern “ __ __ | ” generates monolineal designs using several of the wallpaper pattern groups, while some groups of a small number of walls have been less successful. Of course the wall pattern and the symmetry group alone are not the only defining features for a sona-style pattern; the spacing between the fundamental regions of these groups of walls is equally important. Because of the wide variety of such options, however, student experimenters have many ways to find interesting, symmetric sona; sona that would likely be appreciated by the Chokwe artists themselves. Some of the designs found by students are shown in figure 7. Each of these designs applies to a wide variety of different sizes of rectangles, and each of them meet all of the conditions described above for the type of sona which the Chokwe construct.



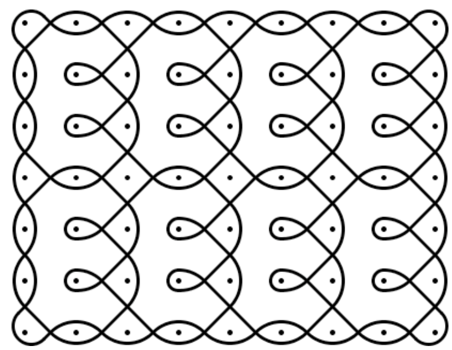
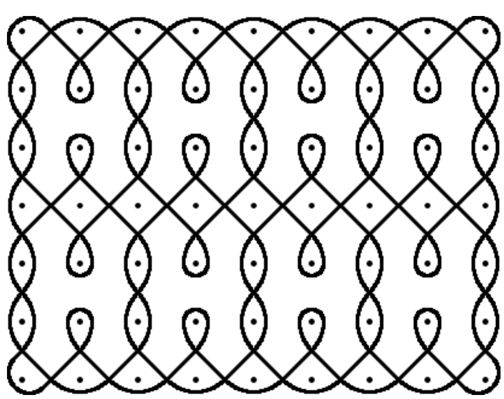
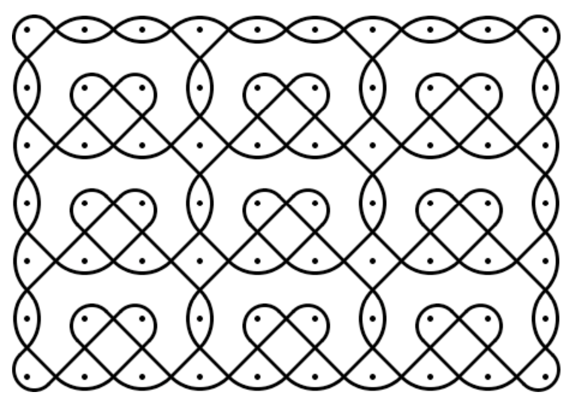


Figure 7: *Sona with wallpaper symmetry, designed by students in the author's class. Clockwise from upper left: by Amy Davis, Paige Witmer, Hannah Edwards, & Mike Whitmore.*





The Sona Program

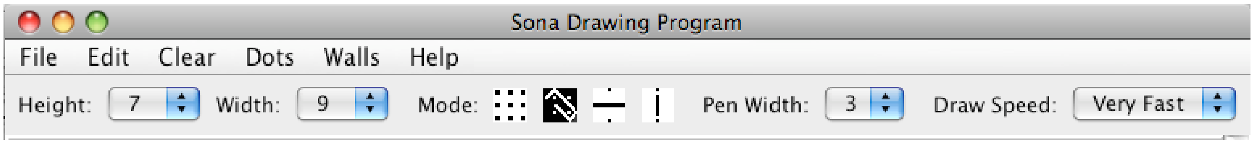


Figure 8: *The main menu bar and tool bar for the Sona Drawing program.*

The sona program allows the user to place rectangles of dots in a grid on the screen, and construct more complicated layouts from unions of such rectangles. The user selects the size of rectangles to add (via the “Height” and “Width” buttons), or 1 X 1 “rectangles” to add individual dots. The “mode” tools allow the user to construct and draw the sona. Figure 8 shows the four tools: The “dots rectangle,” “draw,” “sona wall,” and “Celtic wall” tools. (The Celtic wall tool is under development; it allows the user to draw Celtic knot designs as well as sona drawings.) Choosing the “sona wall” tool allows you to place walls by clicking within the grid. The “draw” tool then initiates the process of drawing a sona line from a user-specified point, in a user-specified direction. If a sona configuration requires multiple lines to complete the sona, each successive line will be drawn in a new color (within limits). The scale of the drawing, the pen width, the speed of the drawing, and whether walls are shown or hidden are each adjustable by the user. Designs can be printed or saved to a file, e.g. to be turned in to the instructor. The instructor can use input files to demonstrate interesting sona, without constructing them in real time, or as starting points for student investigations. The program is written in Java, hence should be fully cross-platform. It is freely available from <http://math.beloit.edu/chavey/Sona>. It has been tested on Macintosh, Windows XP, and Linux systems. (Windows Vista seems to cause problems that have not been resolved as of this article.)

Summary

Sona are useful designs to show students at many levels some of the interactions between mathematics and patterned artwork. They allow students to do both directed experimental investigations, or more open-ended investigations. Explorations topics include common divisors, the Euclidean algorithm for common divisors, and implementations of many types of symmetry groups. The students are challenged to find patterns that would be appreciated by the Chokwe artists, while using the mathematical properties to direct their explorations. Many years of experience by the author, and others, with the software used to construct these sona show the topic to be interesting and exciting to students of various ages, including some who have discovered original results through its use. We will demonstrate several of these explorations in the process of showing the functions and capabilities of the Sona drawing software.

References

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[2] D. Chavey, *Symmetry Groups of Chokwe Sand Drawings*, to be submitted to J. of Math & Art.

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[6] S. Jablan, *Mirror Curves*, < http://members.tripod.com/~modularity/mirr.htm>.