Spirograph Patterns and Circular Representations of Rhythm: Exploring Number Theory Concepts Through Visual, Tangible and Audible Representations

Susan Gerofsky[1], Francisco Gomez[2], David Rappaport[3], Godfried Toussaint[4]
[1]University of British Columbia, Vancouver BC susan.gerofsky@ubc.ca
[2]Polytechnique University of Madrid, Madrid, Spain, fmartin@eui.upm.es
[3]Queen's University, Kingston ON daver@cs.queensu.ca
[4]McGill University, Montréal, PQ godfried@cs.mcgill.ca

Abstract

This paper offers connections between representations in a number of multisensory modes – visual, tactile and audible/musical – that explore a single underlying conceptual structure of periodicity and factorization. A number line is explored in visual, musical and tactile modes with regard to the relationship between regular periodic rhythms; then spirographic geometric patterns and the visual and musical patterns produced by the use of circular representations of musical rhythms are used to develop initial intuitions and observations. Throughout the process, the concept of conceptually-related multiple (multisensory, multimodal) representations is invoked as a way for mathematics learners to feel, hear and see mathematical concepts in depth.

Introduction

Mathematics has special status in terms of theories of embodied knowing because it seems to be the most abstract and conceptual of all fields, ostensibly having little necessary connection to physical experience – or at least, it has come to be considered so in modern times. Historically, mathematics originated with the need to solve very specific, practical problems [9], using a mode of thought essentially visual and kinesthetic. This approach predominated in mathematics until the Enlightenment, when abstraction, conceptualization and formal notation gradually displaced sensory approaches to mathematical reasoning and teaching. By the end of 19th century, mathematics had become highly formal and abstract, and this change of mentality has affected the way mathematics is taught at all levels. We may consider the Bourbaki group as being the greatest representative of abstraction and formalization in mathematics (see [1] and the references therein). While their idea of introducing utmost rigour in the mathematical practice is generally accepted, there has been criticism regarding its introduction at the early levels of mathematical education. Common criticisms to Bourbaki approach are its disregard of discrete aspects of mathematics (algorithms, combinatorial characterizations), its lack of emphasis on logic, and excessive emphasis on exact problem-solving versus exploratory, intuitive heuristics [23].

However, recent work in cognitive science and mathematics education has begun to show that mathematics, far from being disembodied and wholly cerebral, has both roots and expression in bodily knowing [2, 7, 8, 10, 14, 15, 16, 17, 18, 19, 20, 21, 22, 29]. Mathematics education is at the intersection of embodiment and abstraction, physicality and symbolization, where systems of abstract conceptualization are in the process of development. Multisensory, multimodal explorations and the use of physical objects as material anchors and 'tools for thinking' give insight into the details of this cognitive development [11]. Explorations of multimodal possibilities for the embodiment of abstract mathematical structures offer the basis for innovations in the teaching and learning of mathematics. The mathematical explorations introduced below would be appropriate for students learning basic concepts related to fractions, divisibility and factorization; these explorations could be adapted to suit learners and fit curriculum from Grade 5 (age 10) to Grade 10 (age 15) in North American school systems (and the equivalent elsewhere).

Exploring periodicity with rhythms on a number line

Imagine a number line on a long sheet of paper, with numbers starting from zero, increasing in unit increments. If we were interested in exploring the relationship of any two or more numbers, we could use the number line as a way of representing the multiples of those numbers and their relationship to one another, visually, sonically and kinesthetically. For example, if we wanted to explore the relationship between the numbers 3 and 4, we could mark the multiples of 3 on the number line with one symbol or colour, and the multiples of 4 with another one.

Counting up three at a time, we would obtain the numbers 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36,…; counting four at a time, we would get 4, 8, 12, 16, 20, 24, 28, 32, 36,… . Now imagine a group of three students working with this marked-up number line as if it were a musical score. One student might play the role of time-keeper or metronome, and count aloud steadily ("one, two, three, four,…) while tapping successive numbers on the number line. The second student could clap hands (or alternatively stamp a foot or make a sound on a musical instrument) whenever the timekeeper reached a multiple of three. The third student could do the same every whenever the timekeeper reached a multiple of four. If we represent the second student's clap as **S** and the third student's clap as **T**, the resulting sound pattern up to the number 24 could be represented as follows:

S1	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***	***11***	***12***
S2			**S**			**S**			**S**			**S**
S3				**T**				**T**				**T**
S1	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
S2			**S**			**S**			**S**			**S**
S3				**T**				**T**				**T**

Figure 1: *Charting the relationship between a 3- and a 4- rhythm through two periods.*

A pattern begins to emerge. First of all, the second student (counting by threes) and the third student (counting by fours) clap together on the numbers twelve, twenty-four, and continuing the pattern, every multiple of twelve. I would argue that this gives a compelling, embodied demonstration of part of the concept of LCM (Least Common Multiple). Students can feel and hear the necessity of 3 and 4 coinciding at 12. Rather than learning a rule that is applied without sense-making ("if the two numbers are relatively prime, multiply them together to find the LCM"), students would have visual, tactile and auditory sensory representations that demonstrate the least common multiple of two numbers. (Note that Klöwer [4] suggests a similar exercise for a slightly different purpose; Klöwer suggests that percussion students learn rhythmic independence experientially by simultaneously clapping one rhythm with hands and stamping another with feet.)

What is more, it is not only the LCM that is demonstrated, but also the periodically recurring patterns that relate the multiplicative patterns of these two numbers. Going back to the figure, students can start to notice that at the beginning, the third student claps one beat later than the second student (since four is one higher than three). At the next iteration, Student 3 claps *two* beats later than Student 2, and at the following iteration, S3 claps *three* beats later than S2 – at which point the two students' claps coincide, because three beats after one of S2's claps comes the next of S2's claps.

The same pattern repeats over the next twelve beats, from numbers 13 to 24 (or *1* mod *12* to *0* mod *12*) and it can be seen, heard and felt that the pattern will continue its periodic repetition indefinitely. Students can begin to feel, hear and make claims of a more profound *understanding* of the multiplicative relationship between 3 and 4.

Adding a tactile element: rolling a polygonal "wheel" to explore rhythms

A related additional tactile/kinetic representation of the periodicity of 3 and 4 can be made using the same number line. Say the original number line is marked with hatch marks 5 cm apart. Groups of students could be instructed to cut out an equilateral triangle and a square, each with base 5cm, using compass and straightedge constructions and heavy cardboard. The vertices of the triangle and the square could be numbered sequentially counterclockwise (from one to three or four), with the highest-numbered vertex marked in a distinctive colour. If the number line is laid flat on a tabletop or the floor, students can then "roll" the triangle and square simultaneously or sequentially, clockwise along the number line, with vertices hitting the number line on the numbered hatch marks. The effect would be like rolling a square and a triangular wheel along a numbered path. The highest-numbered vertex (3 or 4) will hit the number line at exactly the same spots that have been marked for clapping – that is, the multiples of 3 and 4. (Note that a similar effect, and one that is easier to roll, could be achieved by constructing circles of circumference equal to the distance of 3 or 4 hatchmarks respectively, but that this would have the effect of losing the visual emphasis on 'three' and 'four' that is so obvious when working with the triangle and square – and the construction of circles with precise circumferences is not trivial.)

Students can use tactile, kinetic materials that they produce themselves to demonstrate the logic of periodic multiplicative patterns of 3 and 4, including their LCM. Patterns with numbers other than 3 and 4 can be explored using other regular polygonal "wheels" (pentagons, hexagons, etc.). Note that if a heptagon were desired, then methods other than classic straightedge-and-compass would have to be used – see < http://mathworld.wolfram.com/Heptagon.html> for a reference to knotting methods for this construction.

Exploring the rhythms of related spirographic patterns

Yet another visual/tactile representation of the periodic relationship between two numbers is offered by the toy originally called the Spirograph (and now produced as a dollar-store toy). (A "spirograph algorithm" and the use of spirograph equations and graphics to solve higher-degree polynomials are illustrated at < http://documents.wolfram.com/mathematica/Demos/Notebooks/Spirograph.html>).

A spirograph is a geometric drawing toy made up of plastic gears and shapes (for example, rings and oval bars), all fitted with interlocking teeth. A selected shape can be pinned or taped to a sheet of paper and one of several different-sized gears selected to be rolled along the inside or outside of the shape. A pen point slotted into a hole near the edge of the moving gear traces out a curve on the paper.

A virtual Java-based spirograph is available at: <http://wordsmith.org/anu/java/spirograph.html>, and the reader may wish to visit this site to help visualize the scenarios described in the following paragraphs. A related interactive rhythm/drumming site at <http://www.philtulga.com/unifix.html> may also be of interest here.

Consider just the case of a hollow ring with a smaller gear traveling around its interior as an example. (Other configurations of the spirograph produce similar results in the terms discussed here.) The number of teeth on the inside of the ring and the number of teeth on the smaller gear determine the geometric pattern made by the moving pen.

For example, to take a simple case, imagine a hollow ring with twelve gear slots numbered one to twelve like a clock, and a smaller gear with four teeth. If the tip of a pen is placed in a slot near the edge of the gear as it is rolled along the gear slots on the inside of the hollow ring, starting at 12, it will trace a vertex at each of 4, 8 and 12; the next place the pen will "hit" will be at 12 again, and the pattern will repeat. No matter how many iterations are carried out, and no matter what the offset of the pen placement on the gear, the pattern will have the same three vertices, since 4 is a factor of 12, and 12/4 = 3. (There are three other possible initial placements for the pen tip – at 1, 2 or 3 – which will yield congruent figures with three vertices, offset by one, two or three slots. Figure 4 shows this kind of pattern as part of a brief discussion of necklaces.)

If we continued to use the hollow ring with twelve gear slots, but selected a gear with, say, 10 teeth (a number not a factor of 12 nor relatively prime with 12, but sharing a factor of 2), the pen would trace out a figure with 12/2 = 6 vertices – one vertex each at 10, *20* mod *12* = 8, *30* mod *12* = 6, *40* mod *12* = 4, *50* mod *12* = 2, and *60* mod *12* = 0 (ie. 12). Notice that, while the small gear with 4 teeth traced all four vertices in one revolution of 2 pi radians, the small gear with 10 teeth took 5 revolutions (ie: 10 pi radians) to trace all its 6 vertices, ending up at the top of the "clock" at the zero mark (also called 12).

A small gear with a number of teeth relatively prime to 12 will trace out a figure with vertices at all 12 marks around the "clock". The number of revolutions needed to trace out all 12 vertices will vary. For example, if the small gear had 5 teeth, it would take 5 revolutions (10 pi radians) to complete all twelve vertices (at 5, 10 [end of first revolution], *15* mod *12* = 3, *20* mod *12* = 8 [end of second revolution], *25* mod *12* = 1, *30* mod *12* = 6, *35* mod *12* = 11 [end of third revolution], *40* mod *12* = 4, *45* mod *12* = 9 [end of fourth revolution], *50* mod *12* = 2, *55* mod *12* = 7, and *60* mod *12* = zero or 12 [end of fifth revolution]). Similarly, a small gear with 7 teeth would take seven revolutions to complete all 12 vertices, and one with 11 teeth would take eleven revolutions to complete all 12 vertices. (In fact, since 5 and 7 are co-prime to 12 they act as "generators"-- that is all numbers 0 .. 11 can be generated by a product of *7* mod *12*, or a product of *5* mod *12*.)

Here, in chart form, is the pattern for all the numbers of small gear teeth < 12 for a hollow ring with 12 gear slots:

# of teeth	Relative to 12	Number of vertices	Number of revolutions	Outer ring/inner ring
1	A factor	12	1 (2 pi radians)	12/1
2	A factor	6	1 (2 pi radians)	12/2 = 6/1
3	A factor	4	1 (2 pi radians)	12/3 = 4/1
4	A factor	3	1 (2 pi radians)	12/4 = 3/1
5	Relatively prime	12	5 (10 pi radians)	12/5
6	A factor	2	1 (2 pi radians)	12/6 = 2/1
7	Relatively prime	12	7 (14 pi radians)	12/7
8	Has shared factor(s)	3	2 (4 pi radians)	12/8 = 3/2
9	Has shared factor(s)	4	3 (6 pi radians)	12/9 = 4/3
10	Has shared factor(s)	6	5 (10 pi radians)	12/10 = 6/5
11	Relatively prime	12	11 (22 pi radians)	12/11

Figure 2: *Charting spirographic patterns.*

It is easy to generalize these results to any hollow ring with M gear slots and any smaller gear with n teeth (where $n < M$, and n, M are whole numbers):

> Given M/n in its most reduced form, the resulting figure has M vertices and is produced by n complete revolutions (or in $2n$ pi radians).

Thinking through this general result, there are three cases possible:

a) if n is a factor of M, M/n will reduce to a denominator of 1. The figure will have M/n vertices, produced by one complete revolution.
b) if n and M are relatively prime, M/n will be irreducible. The figure will have M vertices, produced by n complete revolutions.
c) if n is not a factor of M, but neither are the two numbers relatively prime, then M/n will be reducible, but the denominator will be greater than 1. The reduced form of M/n will give the number of vertices as its numerator and the number of complete revolutions necessary to complete the figure as its denominator.

Circular representation of musical rhythms

Observations about periodicity and the relative factorization of two whole numbers, (demonstrated above using a number line and periodic clapping, marking, or rolling of regular polygons, or with a spirograph, using graphic images) can also be experienced and explored using circular representations of musical rhythms.

Toussaint [25, 26, 27] has explored relationships between musical rhythms and geometry using circular representations of musical rhythms. I will borrow Toussaint's representation of rhythms here to offer a further, musical and auditory means of exploration of periodicity and factorization complementary both to linear and to visual representations of the same phenomena. (Note that although Toussaint independently invented the circular representation of rhythms, he later discovered a similar representation used by Persian scholar Safi Al Din in the year 1250 [28, 31]. A similar idea, representing the notes of a scale by a polygon, appears in a Krenek's paper [13] published in 1937, and earlier in Vincent's 1862 publication [30].)

Toussaint describes his circular representations of musical rhythms in terms of a clock with only a rapidly-sweeping second hand [27]. The clock may have any number of numbers or hatchmarks on its face, depending upon the number of beats in a measure of the musical rhythm in question. The numbers marked on the face of this imagined clock correspond to the number of gear slots in the outside hollow ring of the spirograph in our earlier example, but where the gear slots represented a physical mechanism made to produce a graphic pattern, the numbers on the musical rhythm 'clock' represent evenly-spaced rhythmic 'beats' in a musical pattern.

It is interesting that these two different kinds of 'clock charts' (the spirograph and the circular representation of musical rhythm) can use the same form to represent widely divergent phenomena which might share similar structures of pattern and periodicity. The number line referred to earlier in this paper is also homologous to these 'clock charts'; imagine choosing a number from the number line and then wrapping the number line into a loop of concentric circles, all with that circumference. The resulting circle would look very much like one of the clock diagrams used to represent musical rhythms or the patterns of the spirograph.

Let us return to the 'clapping on a timeline' example given earlier in this paper. In this example, one student marked out points on the timeline that were evenly spaced three apart, and the other student marked points space four apart, and then clapped their two rhythms to a third student's 'metronome' beat. It was found that every twelve beats, the two students would clap at the same moment, and that the 'four-beat' person would clap at first one, then two, and then three beats after the 'three-beat' person (at which point they would coincide). The pattern would repeat indefinitely with a period of twelve beats.

Converting the timeline to a rhythm circle with a circumference of 12 (marked out as a clock with twelve hatchmarks, but numbered for convenience with the '1' at the top of the clock since, by convention, music time is beaten starting with a downbeat on '1'), the four-beat and three-beat patterns would look like this:

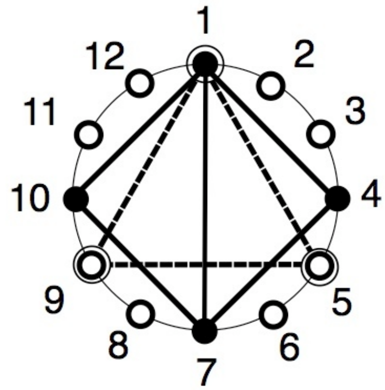

Figure 3: A 3- and 4-rhythm on a clock of 12

Note that the 'every-three-beats' rhythm lands on beats 1, 4, 7, 10, and then 1 again, yielding a square with four vertices; the 'every-four-beats' rhythm lands on beats 1, 5, 9, and then 1 again, yielding an equilateral triangle with three vertices. This effect is homologous to the pattern yielded by rolling an equilateral triangle and a square along the number line – but when the line is 'rolled up' into a circle, the triangular gear yields a square, and the square gear yields a triangle! On the number line, we used a triangle and a square with bases of equal length. Here, the square created within the circle of twelve has a base length of three units, while the equilateral triangle has a base length of four units. This allows space to contemplate the fact that 3 and 4 are complementary modulo 12.

Another interesting observation that can be made using three-beat and four-beat patterns on a circle of twelve are the possible so-called 'necklace patterns'. Toussaint describes a necklace as "the inter-onset duration interval pattern that disregards the starting point in the cycle…If a rhythm is a rotated version of another we say that both belong to the same *necklace*." [26, p. 5]

The following figures give all the possible patterns in the necklace of two-beat intervals on a rhythmic clock of six (figure (a)) and of three-beat intervals on a rhythmic clock of twelve. The figures created are rather beautiful star shapes made up of overlaid discrete regular polygons (equilateral triangles in figure (a) and squares in figure (b)).

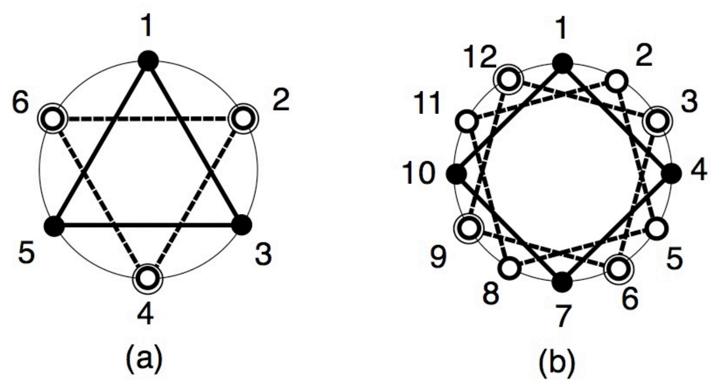

Figure 4*: Necklaces of a 3-rhythm on a clock of 6 and a 4-rhythm on a clock of 12*

It is important to note that, as these are images of rhythmic patterns, the rhythms are meant to be both sonic and visual. The visual representations of these rhythms can be treated as structurally equivalent to the *sonification* ("the use of nonspeech audio to convey information" http://www.icad.org/websiteV2.0/References/nsf.html) of the pattern. So these visual representations are meant to be sonic and kinesthetic – meant to be clapped, drummed, played on other musical instruments, stamped, even danced [3, 4, 5, 6, 29].

A different kind of star pattern can be created in sonifications of regularly-spaced rhythmic patterns. In a pattern exactly parallel to the one discussed previously with reference to the spirograph, the number of a rhythmic 'clock' may be overlaid with a regular 'beat' that is (a) a factor of the number of the clock, (b) relatively prime with regard to the clock, or (c) one which is neither relatively prime nor a factor, but which shares one or more prime factors with the number of the clock. The clock of twelve overlaid with a beat of three or four, and the clock of six overlaid with a beat of two are examples of (a), beats which are factors of the clock (since 3 and 4 are factors of 12, and 2 is a factor of 6).

Here, in contrast, is the visual representation of a beat of 3 on a clock of 7:

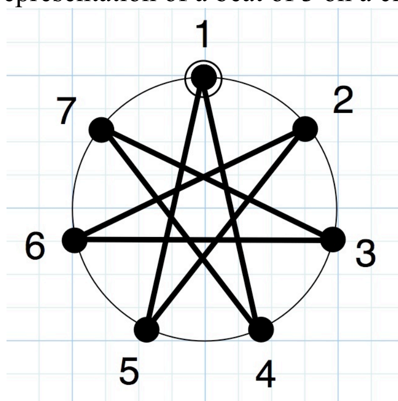

Figure 5: *A 3-rhythm on a clock of 7, forming a star polygon*

Since 3 and 7 are relatively prime, the beats fall on the clock at 1, 4, 7, *10* mod *7* = 3, *13* mod *7* = 6, *16* mod *7* = 2, *19* mod *7* = 5, *22* mod *7* = 1. The resulting figure is an aesthetically-pleasing regular star polygon with 7 vertices – a figure very much like those produced on a spirograph. As predicted by our earlier observations of the spirograph, the pattern of M/n with M = 7 and n = 3 yields a figure with 7 vertices that takes 3 full rotations to complete. In musical terms, this translates to a pattern that has salient beats that eventually hit all seven beats of the 7-beat bars, and repeats after every three full bars or measures.

Conclusion

The exploration above of visual, sonic and kinesthetic representations of aspects of multiplicative relationships and factorization offers a starting point for the development of innovative teaching and learning activities in mathematics education. It is hypothesized that students engaged in making multiple, multimodal representations of mathematical relationships as well as learning to make sense of equivalent structures represented in different sensory modes develops deeper understandings of those mathematical structures and patterns [24]. Here we have presented an approach to teaching concepts of periodicity and factorization that incorporates visual, tangible and audible representations.

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