Spidronised Space-fillers

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Abstract

Saddle polyhedra have faces that are skew polygons, with edges that do not lie in one plane. The surface of a face can be undefined [1], a minimal surface [2], triangulated [3], or filled using a spidron nest [4,5]. Identifying circuits in three dimensional periodic networks of vertices and edges [6] with saddle faces generates space-filling saddle polyhedra, described in [2]. We consider these space-fillers, and by extending the concept of a spidron so it can be applied to the faces create forms that are visually interesting, both as individual polyhedra and in aggregations.

Spidrons and Spidron Nests

A spidron was originally defined as a particular infinite set of triangles that tiles the plane, and a spidron nest as a combination of semi-spidrons that form a hexagon [4]. This idea can be extended in a fairly natural way to work with any regular polygon with an even number of sides, but there are problems if the polygon is not regular (see later for more detailed discussion).

Throughout this paper a polygon is considered, as in [1], as a closed circuit of edges, meeting at vertices. In particular a polygon is considered to be distinct from its interior. A polyhedron is considered to be a surface distinct from its interior. Only two faces of a polyhedron meet at any edge, but they can share more than one edge.

Dániel Erdély discovered that the interior of a hexagon divided into an infinite series of similar rings, each consisting of six equilateral and six 120° isosceles triangles (a spidron nest), can fold so that the hexagons become non-planar. By noticing that a cube can be dissected along a skew hexagon he was then able to construct a space-filling octahedron [7]. If the skew hexagons are simply triangulated then the polyhedron that is generated is the first stellation of the rhombic dodecahedron, which is a well-known space-filler [8]. If a minimal surface is used it corresponds to Pearce's #40, figure 8.55 in [2], which is used in his space-filling #4, illustrated in his figure 8.68.

In fact spidron nests can be folded in many different ways, since each ring can turn either clockwise or counter-clockwise. Usually the choices are made in a consistent way so that the spidronised polygon appears as a many-armed spiral in relief. However the choices are made, in a space-filling, faces that coincide must correspond, so that a clockwise ring matches a counter-clockwise ring. This has important consequences.

In order to make spidronised versions of the faces of the saddle polyhedra used by Pearce we need to consider their edges, which are usually skew polygons. As long as the skew polygon is regular (equilateral and equiangular) it is not too difficult to construct a corresponding spidron nest. In fact, within certain limits, there are two degrees of freedom, which can be thought of as the angles of one of the triangles in the dissection.

Unfortunately it is more usual to need skew polygons that are not regular, even to make a single polyhedron before the constraints of space-filling are considered. There is an obvious construction that generates visually satisfying forms, but in general the spidron nests produced cannot fold.

Start with a polygon, which may be skew. Usually it will have some rotational symmetry, so there is an obvious centre, but it may be necessary to make some more or less arbitrary decision about which point to take as the centre. Make a copy of the polygon, scale it down by some factor towards the centre and rotate it by some angle. Triangulate the region between the two polygons by joining every point on the original (outer) polygon to the images of its two neighbouring vertices. Apply the similarity transformation (scale + rotation) to the resulting surface, which, by construction, will fit inside. The transformation can be repeated indefinitely to give a series of rings that converge towards the centre. The set of images of any point lies on a logarithmic spiral.

Constructing Space-filling Polyhedra

The faces of any polyhedron can be spidronised using this construction, and because the (arbitrary) rotation around the axis can be in either direction, there are clockwise (CW) and counter-clockwise (CCW) versions of the nests. For a single polyhedron there is no restriction on how these versions are chosen, but in a space-filling, faces that meet must match. Looking from the outside of the polyhedra a CW face is matched by a CCW face. We want to choose these orientations so that the number of different spidronised polyhedra is minimised, if possible with a single spidronised form for every copy of a polyhedron in a space-filling. This is often quite difficult to achieve, and sometimes impossible.

Pearce [2] lists 42 space-filling systems using a total of 54 polyhedra with 34 different polygons as faces, but he acknowledges that this list is not exhaustive. Figure 1 shows all these faces, with the skew polygons spidronised. For each space-filling a *translational unit* can be identified that generates the complete space-filling by translations only. The translational unit consists of one or more *basic repeat units*, which are the smallest aggregations of spidronised polyhedra that fill space by themselves. Generally the numbers of polyhedra in the repeat unit are given in space-filling ratio listed in Pearce, but there are circumstances when twice as many are needed because of the requirement for equal numbers of CW and CCW versions of a face, for example in a space-filling that uses a single type of polyhedron with an odd number of faces.

The situation that can occur when (unspidronised) faces are enantiomorphic is rather less obvious. An enantiomorphic face is a three dimensional object, rather like a helix, so that right-hand, R, and left-hand, L, forms can be interchanged by reflection, but in no other way. The process of spidronisation is essentially two dimensional, and CW and CCW forms can be interchanged by a 180° flip. Since R faces must meet with R faces (and L with L) we can consider each type separately, so a polyhedron with an odd number of R faces behaves like a polyhedron with an odd number of identical faces, and the requirement for an equal number of CW and CCW versions implies an even number of different spidronised forms of the polyhedron.

Some space-filling polyhedra listed in Pearce have faces that are not enantiomorphic, having mirror symmetry, but they are two-sided, in the sense that their appearance is not conserved under a 180° flip. Such polygons can be spidronised in two ways, either with the “A” side CW, or the “B” side CW. Since in a space-filling the “A” side of a polygon must meet the “B” side of its mate, all that is needed is to use the same spidronisation throughout, and CW will always meet CCW.

All of the saddle polyhedral space-fillers in Pearce can be constructed applying these general principles, but they can have different consequences. Two examples illustrate the main points.

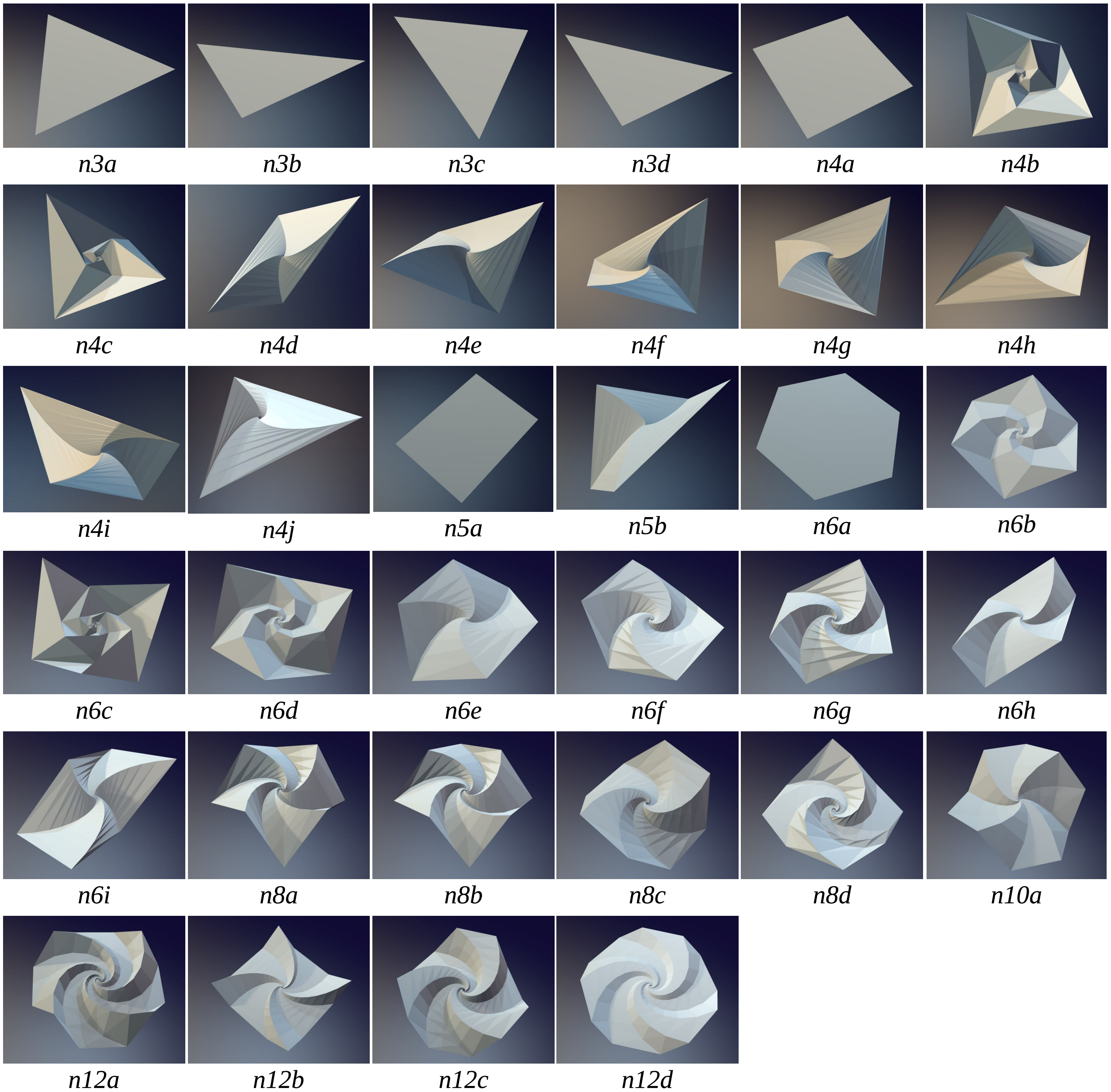

Figure 1: *Total set of 34 nests for spidronised space-filling polyhedra*

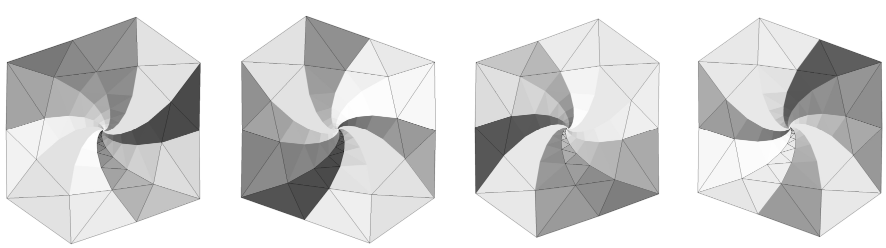

Figure 2: *Two sides and the chiral versions of the decagonal spidron-nest (n10a)*

Example 1 - The Decatrihedron (The Triamond Space-filling)

Pearce's first space filler has three skew decagonal faces, hence his name of *decatrihedron*. The decagons are circuits in the *triamond* lattice [9], hence its classification as [10, 3]. Although the decagons are equiangular, with an included angle (G-angle) of 120°, they are not regular, so the spidronised version is rather different from previously published examples (Figure 2).

The polyhedron faces come very close to each other so some care is needed in choosing the parameters (scale factor and rotation angle) to construct the spidron-nest so that intersections are avoided.

Since the polyhedron has an odd number of faces two different spidronised forms are needed, and the basic repeat unit consists of two polyhedra with four external faces (Figure 3). There are three ways to achieve this. The two external CW faces can be from the same polyhedron, or there can be one face from each. In the latter case moving between faces of the same type (CW or CCW) involves a screw rotation of 90° along a helix that can be of either sense. Of course there is further variation if we consider the two possible alternatives for the internal faces. As Figure 2 indicates the decagon is chiral, so it has two enantiomorphic forms, either of which can be used to construct the polyhedra, so the final space-filling can be of two forms.

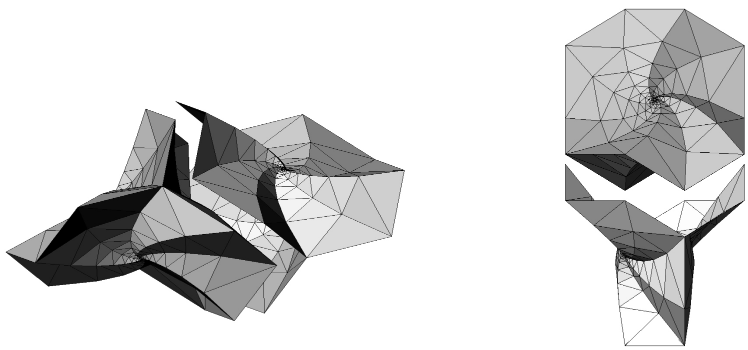

Figure 3: *Joining two spidronised decatrihedra (one of the three possibilities). Middle faces omitted.*

Example 2 - A More Complicated Example

One of the most complicated examples is Pearce's #41, which has a basic repeat unit consisting of ten polyhedra. Finding the correct orientation for each spidron-nest is far from easy. In one sense the internal faces are easier since they could be oriented randomly and the basic repeat unit would still work, but they should be chosen so that each polyhedron appears as only one spidronised form. There are 16 such internal face to face meetings. The basic repeat unit has 60 outer faces, and there are seven different directions of translation to neighbouring repeat units.

Four of the nests are of the "mirror" type with "A" and "B" sides described above. This makes things slightly easier, since the unspidronised structure determines the orientations of the spidrons.

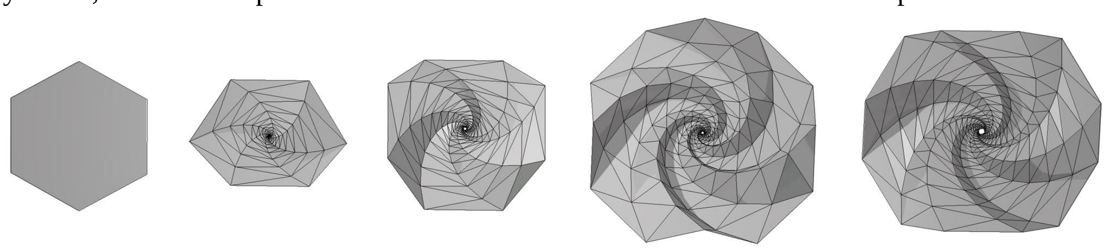

Figure 4: *The spidron nests used in making Pearce's space-filling #41.*

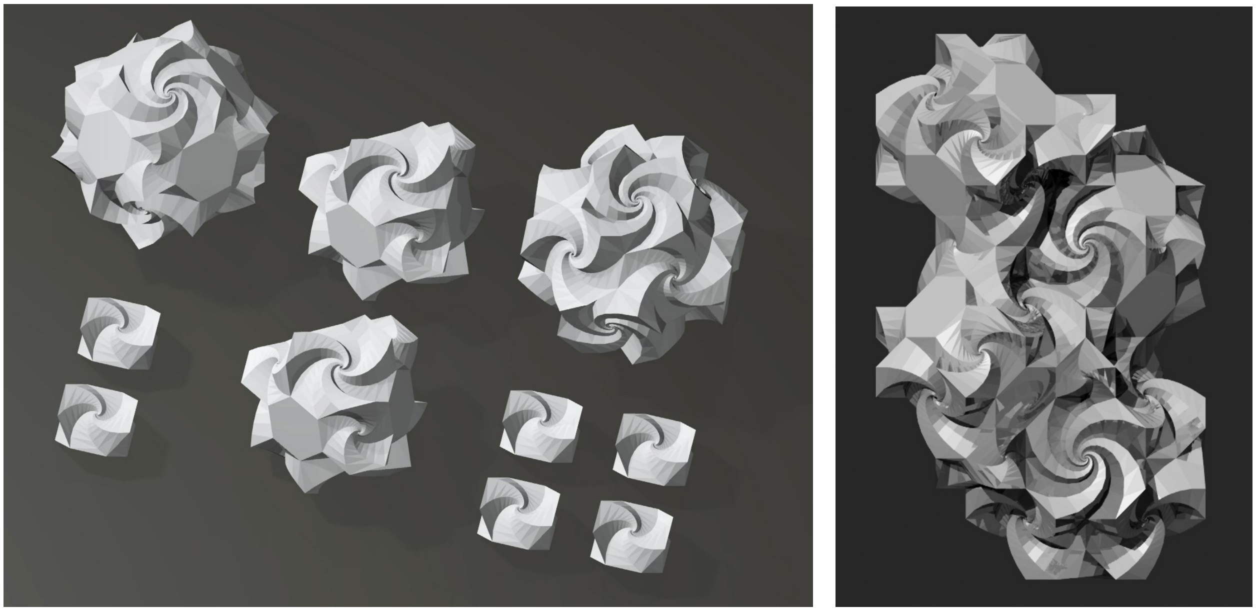

Figure 5: *The spidronised polyhedra used in a basic repeat unit of #41, and the assembled unit*

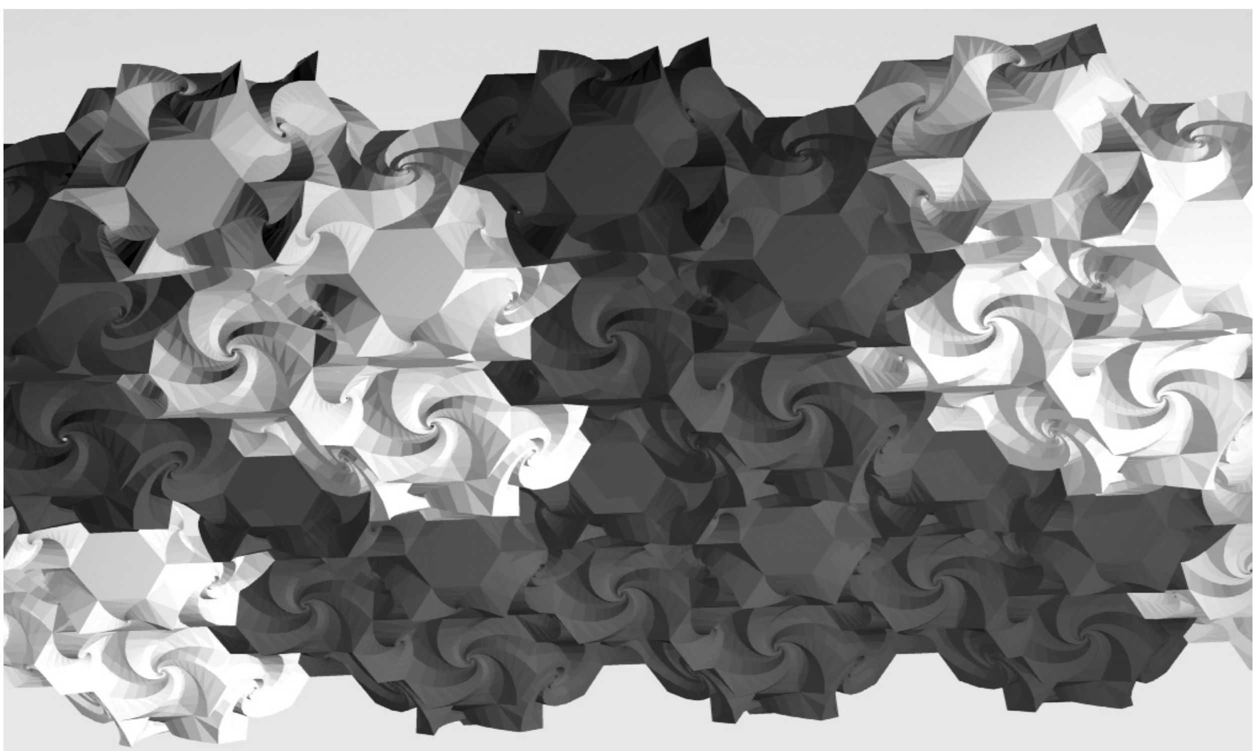

Figure 6: *A piece of Pearce's space-filling #41, coloured to show the basic repeat units*

Tables

The following tables summarise all 34 nests and 42 space-fillings described by Pearce, and provide details of spidronised versions. There are two errors in Pearce's space filling ratios which have been corrected *(marked with *)*.

Nest code	Used in polyhedra	Used in spacefillers	Polygon	Group	Symmetry	G-angles	Zome Code
n3a	52 cubocta	36	3-gon	flat	3-fold	3x60	GGG
n3b	02 09 26	24 29 32	3-gon	flat	no symm	90;54.7;35.3	BG2Y
n3c	04 07 09	08 13 32	3-gon	flat	mirror	54.7;70.5;54.7	YYB
n3d	10	32	3-gon	flat	mirror	45;90;45	BBG
n4a	30 33 34 cubocta	17 20 21 36 37 38	4-gon	flat	4-fold	4x90	BBBB
n4b	12 35 36	03 15 19	4-gon	regular	2-fold	4x70.5	YYYY
n4c	14 43	05 16 42	4-gon	regular	2-fold	4x60	GGGG
n4d	02 15 28	24 27	4-gon	enantio	2-fold	2x(45;90)	BGBG
n4e	03 13 14 26 32 44 51	16 23 26 29 37 39 42	4-gon	mirror	2-fold	2x(60;90)	GGGG
n4f	04 27	06 08	4-gon	enantio	2-fold	4x54.7	BYBY
n4g	03 05 24	18 39	4-gon	mirror	mirror	60;90;90;90	BBGG
n4h	08 20	28	4-gon	mirror	mirror	109.5;54.7;90;54.7	BBYY
n4i	07 16	09 13	4-gon	enantio	no symm	90;54.7;54.7;90	BBB2Y
n4j	09 10 23 39	32 33	4-gon	enantio	no symm	90;45;54.7;54.7	B2Y2BG
n5a	18 22	12 31	5-gon	flat	mirror	90;90;180;90;90	GBBG2B
n5b	21 22 42	30 31	5-gon	mirror	mirror	5x90	BGGB2B
n6a	19 46 53	07 14 25 41	6-gon	flat	6-fold	6x120	6xG
n6b	11 25 38 47	02 10 15 22 34 35	6-gon	regular	3-fold	6x109.5	6xY
n6c	24 43 49	05 18 20 42	6-gon	regular	3-fold	6x60	6xG
n6d	24 40	04 18	6-gon	regular	3-fold	6x90	6xB
n6e	17 30 31 52 53	17 25 36 38 41	6-gon	mirror	2-fold	2x(90;120;120)	6xG
n6f	06 25	22 34	6-gon	mirror	2-fold	6x109.5	no zome!
n6g	18 41	11 12	6-gon	mirror	mirror	6x90	2x(GGB)
n6h	19	14	6-gon	mirror	mirror	2x(90;90;120)	2x(GGY)
n6i	38	10	6-gon	mirror	mirror	2x(70.5;70.5;109.5)	6xY
n8a	32 33 49 51	20 26 37 42	8-gon	mirror	4-fold	4x(60;90)	8xG
n8b	29 35 37 47	15 35 40	8-gon	mirror	4-fold	4x(70.5;109.5)	8xY
n8c	17 46	07 17 41	8-gon	mirror	2-fold	8x120	8xG
n8d	37 45	40	8-gon	mirror	2-fold	2x(90;144.7;109.5;144.7)	2x(GGYY)
n10a	01	01	10-gon	enantio	2-fold	10x120	10xG
n12a	31 34 50 53	21 25 38 41	12-gon	mirror	4-fold	4x(90;120;120)	12xG
n12b	36 48	19	12-gon	mirror	4-fold	4x(70.5;144.7;144.7)	4x(YYG)
n12c	46 50	07 21 41	12-gon	mirror	3-fold	12x120	12xG
n12d	48	19	12-gon	mirror	3-fold	12x144.7	6x(YG)

Table 1: *The 34 nests*

Nest Code: an identifier based on the number of edges.
Used in polyhedra: referred to Pearce's table 8.1, also used in Table 2.
Used in spacefillers: referred to Pearce's table 8.2, also used in Table 2.
Polygon: taken from Pearce.
Group: there are 4 kinds of nests: flat, regular, mirror and enantiomorphic.
Symmetry: taken from Pearce.
G-angles: between adjacent edges (rounded to 1 decimal place).
Zome Code: Zometool [10] struts used to make a model. 2Y means two Yellows in the same line.

Space-filler	Polyhedra	Ratio	Nests classified per type				Symmetry	Factor SF	Outer Nests	Types	Factor Unit
			Flat	Regular	Mirror	Enanthio-morphic					
1	1	1				n10a	E	2	**4**	4	4
2	11	1		n6b			R	1	**4**	4	2
3	12	1		n4b			R	1	**4**	4	6
4	40	1		n6d			R	1	**8**	8	1
5	43	1		n6c n4c			R	1	**10**	4,6	2
6	27	1				n4f	E	2	**10**	6+4	8
7	46	1	n6a		n12c n8c		M	1	**14**	4,4,6	2
8	4	1	n3c			n4f	E	2	**4**	2,2	12
9	16	1				n4i	E	1	**4**	2+2	12
10	38	1		n6b	n6i		M	2	**10**	4,6	1
11	41	1			n6g		M	1	**8**	8	1
12	18	1	n5a		n6g		M	1	**4**	2,2	4
13	7	1	n3c			n4i	E	2	**4**	0,2+2	4
14	19	1	n6a		n6h		M	2	**6**	0,6	2
15	47 35	1 3		n4b n6b	n8b		M	1	**26**	12,8,6	1
16	13 14	1 2 *		n4c	n4e		M	1	**8**	4,4	1
17	30 17	1 2 *	n4a		n6e n8c		M	1	**10**	2,6,2	1
18	5 24	1 1		n6c n6d	n4g		M	2	**10**	8,2,0	1
19	48 36	1 3		n4b	n12b n12d		M	1	**26**	12,6,8	1
20	49 33	1 3	n4a	n6c	n8a		M	1	**26**	12,8,6	1
21	50 34	1 3	n4a		n12a n12c		M	1	**26**	12,6,8	1
22	6 25	1 1		n6b	n6f		M	1	**6**	2,4	2
23	13 44	3 1			n4e		M	1	**18**	18	1
24	15 2	1 4	n3b			n4d	E	1	**8**	8,0	6
25	53 31	1 3	n6a		n6e n12a		M	1	**38**	8,24,6	1
26	51 32	1 3			n4e n8a		M	1	**30**	18,12 or 24,6	1
27	28 15	2 3				n4d	E	1	**14**	8+6	2
28	8 20	4 3			n4h		M	1	**12**	6+6	2
29	13 26	1 4	n3b		n4e		M	1	**14**	10,4	6
30	21 42	2 1			n5b		M	1	**12**	6+6	1
31	21 22	1 2	n5a		n5b		M	1	**8**	2,6	1
32	9 10	2 1	n3b n3c n3d			n4j	E	1	**6**	2,0,2,2	12
33	23 39	3 2				n4j	E	1	**14**	8+6	4
34	6 25 11	1 1 2		n6b	n6f		M	1	**10**	6,4	2
35	47 29 11	1 1 2		n6b	n8b		M	1	**22**	12,10	1
36	52 30 cubocta	1 3 1	n3a n4a		n6e		M	1	**38**	14,6,18	1
37	33 32 13	1 1 1	n4a		n4e n8a		M	1	**12**	4,6,2	1
38	31 34 30	1 1 1	n4a		n6e n12a		M	1	**14**	6,6,2	1
39	13 5 3	3 8 12			n4e n4g		M	1	**24**	0,24	1
40	29 37 45	1 3 1			n8b n8d		M	1	**24**	6,18	1
41	53 46 50 17	1 2 1 6	n6a		n6e n8c n12a n12c		M	1	**60**	12,14,10,10,14	1
42	49 51 43 14	1 1 2 6		n4c n6c	n4e n8a		M	1	**46**	12,12,12,10	1

Table 2: *The Space-fillers*

Space-filler: an identifier, taken from Pearce table 8.2.
Polyhedra: as in Table 1.
Ratio: space-filling ratio as in Pearce.
Nests: classified by type.
Symmetry: minimal symmetry, where E is **”**smaller**”** or lower symmetry than M, so E < M < R < F.
Factor SF: multiplication needed to match nests.
Outer Nests: number of outer faces in the spidronised repeat unit.
Types: numbers of each kind of nest (in corresponding order). A + sign means there are chiral versions.
Factor Unit: multiplication needed to create a translational unit.

Further Work

The similarity transformation method has been used to construct the faces of almost all of the polyhedra considered. This is satisfactory so long as computer images, or models produced by rapid prototyping are adequate, but it would be more convenient, and cheaper, to be able to make the faces from single sheets of material. This means that the faces need to fold, and the behaviour of spidrons as foldable linkages, apart from the regular examples, is at present poorly understood. We do not even know very much about the way regular spidrons behave when folded in non-symmetric ways. Much remains to be discovered.

In order to make progress through a large number of examples we have proceeded by trying to find the smallest aggregation of spidronised polyhedra that will fill space on its own periodically. A different approach would be to start from particular spidronised forms of known space-filling polyhedra, and determine whether they will fill space, and how.

More detail about the 3D structure of edges and their projections will be shown on the a special CD that will be made available during Bridges 2009. It will also contain further Excel tables and coloured pictures and animations of all the spidronised space-fillings.

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