Three Musical Geometries

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Music theorists have devised three measures of musical distance, each giving rise to different geometry. In my talk I will describe these geometries and discuss some of ways in which they are related. My conclusion is that while there are important musical circumstances in which the three geometries converge, this is largely a coincidence—an artifact of the special cases that music theorists habitually consider. Understood more robustly, the three geometries are fundamentally different, and suggest very different ways of understanding music. Unclarity about this fact has led to some interesting twists and turns in recent music theory.

The first measure is based on acoustics. It asserts that the distance between two notes is given by the "simplicity" of the ratio between their fundamental frequencies. Thus, E5 (660 Hz) is closer to A3 (220 Hz) than Bf3 (234.667 Hz) because the ratio 3/1 is "simpler" (in some sense) than 16/15. This conception of musical distance gives rise to discrete lattices such as Euler's *Tonnetz*. As Guerino Mazzola has emphasized, these lattices can also be understood number-theoretically, as embeddings in finite extensions to the field of rational numbers.

The second metric reflects log-frequency distance. According to this conception, Bf3 is much closer to A3 than E5 is, because the absolute value of the logarithm of the ratio 234.667/220 is smaller than that of 660/220. (This conception reflects physical distance on the ordinary piano keyboard.) By extending this measure of distance to collections of notes, it is possible to construct geometrical spaces containing all conceivable chords: for example, log-frequency distance between pairs of notes, modulo the octave, is reflected on a Möbius strip. These spaces are orbifolds, whose description requires ideas from recent geometry and topology.

The third definition of distance is given by the total interval content of a group of notes. According to this metric, the distance between two chords is inversely proportional to the similarity of their "interval vectors" (the multiset of pairwise distances between notes in the chord). Homometric chords are from this perspective musically identical. This metric can be investigated using the Fourier transform—not of acoustical signals, but of the characteristic function representing a chord's note content. As Ian Quinn has shown, the resulting Fourier magnitudes can be used to define six-dimensional geometrical spaces that seem to represent the harmonic "quality" of a collection of notes.

Remarkably, to a rough first approximation, the three models seem to be mutually consistent. For example, major and minor triads are represented by triangles on Euler's *Tonnetz*. The "distance" between two triangles appears to represent the size of the minimal log-frequency distance between the relevant chords. Similarly, distances in Quinn's "quality space" are highly correlated to distances in the orbifolds derived from the log-frequency distance. In other words, despite their different mathematical origins, the three intuitive notions of musical distance are roughly convergent, each reflecting a more fundamental conception of similarity. This has led some music theorists to use geometrical models derived from the first and third conceptions of musical distance to reflect facts about the second.

In my talk I will suggest that this is misleading. The confusion of the three conceptions of musical structure can lead to theoretical difficulties, which can be surmounted only if by rigorously distinguishing the various models. Both theoretically and practically, it is important think carefully about which is best-suited for a particular musical purpose.