Mathematical Secrets of Seven

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Abstract

The number seven is associated with a variety of phenomena, both real and symbolic. However, in the physical world of ornamentation and architecture, it is less widely utilized than other numbers. This paper examines the regular heptagon in some depth and looks at particular characteristics of the number seven itself.

Nearly everyone agrees that all numbers are created equal, that numbers are just numbers, and nothing else. They have no inherent special meaning, and yet they can be used to describe the workings of the universe. Such is the beauty of the science of mathematics.

The fascinating worlds of architecture and ornamentation often make use of regular figures, particularly regular polygons which can be constructed easily and accurately. There is a preference for those figures with even numbers of sides. On the two-dimensional plane, tiling often figures prominently and of course only the triangle, the square, and the hexagon can cover the plane without gaps or overlaps.

Even so, the pentagon appears fairly often with it's association with the Golden Section and a perceived inherent attractiveness. But for a variety of reasons, seven-sided figures do not appear as often in ornamentation as other regular-sided polygons. Even the enneagon (9-sides) is more common although it, too, is relatively rare.

Approximate Construction of Regular Heptagons

The regular heptagon cannot be precisely constructed using only a straight-edge and compass. However, a number of very close approximate constructions have been devised, some of them dating back to antiquity. Figure 1 catalogs five of these which will be explained briefly here. [2], [4], [5]

Logically, by calculating 360/n, we see that the central angle of the heptagon should be 51.428571… degrees. We can determine the accuracy of each approximate method by comparing it's central angle with this number.

a) Draw the radius of the circumcircle. Construct a segment of the same length perpendicular to the radius. Find the point E such that CD touches the circle. OE will be the length of the side of the regular heptagon. Notice, this construction requires a measurement, or a marked straight-edge and is therefore not a classic Euclidean construction.
b) Given an equilateral triangle within the circumcircle, find the midpoint C of one side. Using this length, scribe an arc from the vertex F of the triangle to cross the circle at G. The chord FG will be the length of the side of the regular heptagon. This method was used by Albrecht Dűrer in the early 16th century.
c) The angle whose tangent is 5/4 (51.3402°) is very close to the central angle of the regular heptagon.
d) Beginning with a circle, construct a square enclosing it such that the sides are tangent to the circle. From two adjacent corners, draw quarter circles with the radius equal to one side of the square. Connect the point of intersection of these two quarter circles (I) to one of the corners.

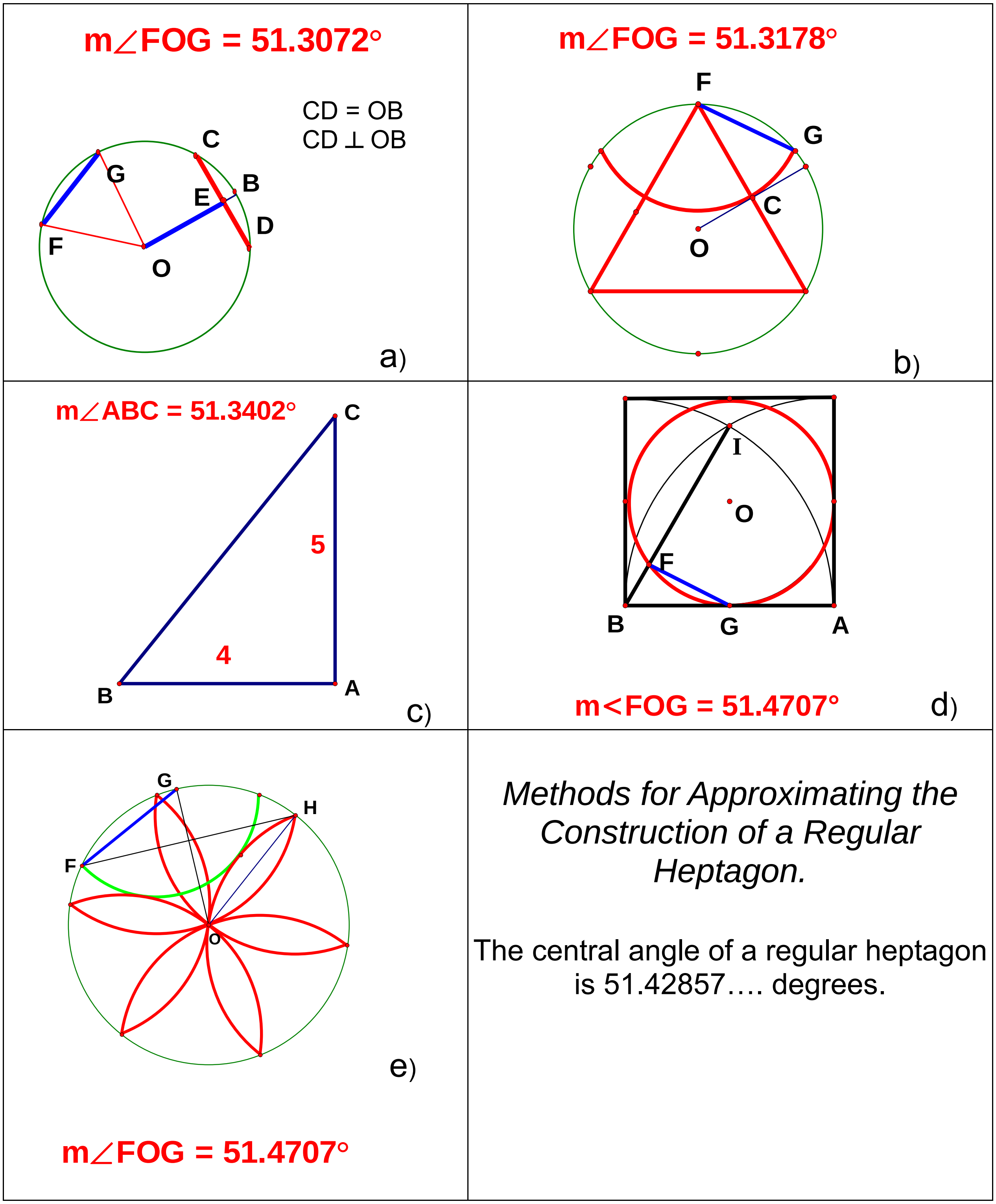

Figure 1: *Methods for approximate construction of regular heptagons*

Call the intersection of this line with the circle, F. Draw a segment from F to the midpoint of the base of the square, G. FG will be the length of the side of a regular inscribed heptagon.

e) Beginning with six arcs drawn to divide the circle into six equal parts, draw an arc from one point that "just touches" the neighboring petals and continues to cross the original circle. At the crossing point F draw a line to the tip of the second distant petal. Bisect this double angle FOH to find point G. FG will be the length of the side of the regular heptagon.

Measurements of a Regular Heptagon

Having constructed an approximate regular heptagon, what do we know about the actual measurements contained within it other than the *central angle* of 51.428571 degrees?

The *vertex angle* can be deduced from the following: Draw seven interior triangles. Each triangle has a total of 180° so there is a total of 7(180)=1260°. Now subtract the central angles (360°) to leave just the total of the vertex angles. All the vertex angles are equal, so they must each be 900/7 or 128.57142°.

The *radii* of the circumcircle and the inscribed circle can be calculated in terms of the length of the side E of the polygon. See the diagram in Figure 2. In that figure a close look at ½ the central angle (ABC) gives an equation for finding the length of the radii.

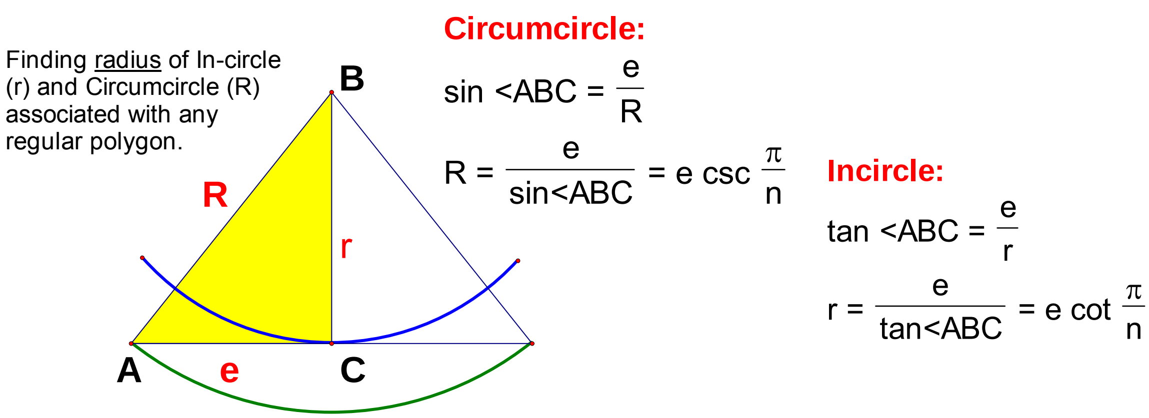

Figure 2: *Finding the radii of the incircle and circumcircle of any regular polygon*

The *incircle* radius (r) of a regular heptagon is e cot (180/7) = (E/2) (2.0765) = 1.0383 E (the length of one edge). The *circumcircle* radius (R) is e csc (180/7) = (E/2) (2.3048) = 1.1524 E.

To find the *area* of the interior of the polygon, first find the area of ΔABC, double it, and multiply by n. For the heptagon, the area of ΔABC = ½ (base)(height) = ½ e e cot (180/7). Doubling this and multiplying by 7 gives $7e^2$ cot(25.7143) = 14.5356 $(E/2)^2$ = 3.6339 E^2 = area of regular heptagon where E is the edge length.

Repeating Decimals and Divisibility by Seven Tests

Moving from geometry to number analysis, we see that an interesting pattern develops when dividing by seven. Division of *any* number by seven yields a repeating string of the *same* digits, 142857….. in the same order, except in cases where the number is a multiple of seven. Three, 6, and 9 never occur.

There is a rather unusual, if inefficient, test that will determine if a number is evenly divisible by seven. Extract the last (ones) digit, multiply it by 5, and add it to the remaining digits to get a new number. Now repeat the test if necessary, until you arrive at a small number which you can determine is or is not a multiple of seven. [3] Here is an example:

945,238; 94,523 + **5x8** = 94,563; 9,456 + **5x3** = 9,471; 947 + **5** = 952; 95 + **5x2** = 105
10 + 25 = 35 which is evenly divisible by 7!

Here is a further technique for testing very large numbers. Divide the number into three digit clusters (as by grouping with commas). Take the first (left-most) cluster, *subtract* the second cluster, to the result *add* the third cluster, *subtract* the forth, and so on. When you are finished if you still have a large number, you can apply the previous technique. By the way, this test also works for the numbers 11 and 13.

Consider the number 784,639,801,295.
784 – 639 = 145 145 + 801 = 946 946 – 295 = 651 which is evenly divisible by 7.

Conclusion

Using the characteristics of seven in an artistic manner provides a special kind of challenge to would-be designers, but the repeatability of digits after division offers one good place to start. Here are two designs using lines of lengths in the same order as the remainder digits 142857… The square offers interesting interior spaces which emphasize the pattern of the digits.

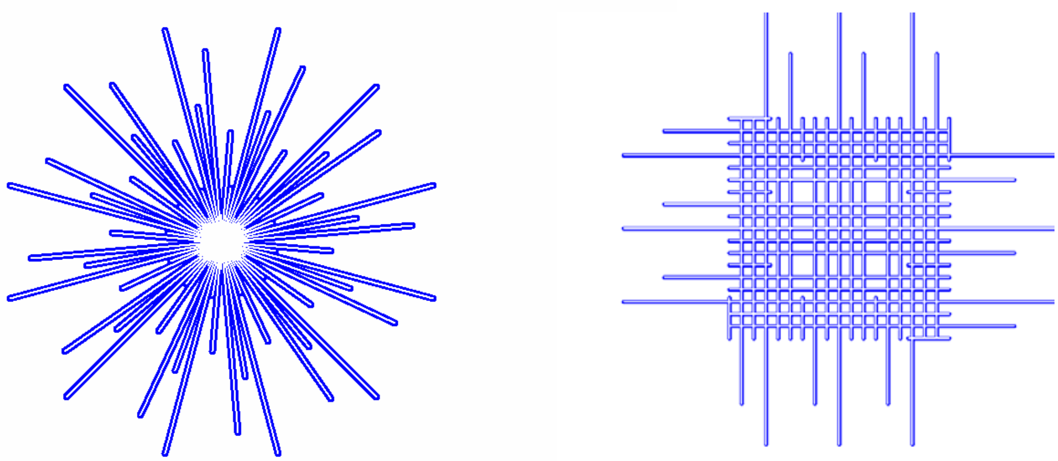

Figure 3: *Two designs implementing the remainder digits 142857 after division by 7*

References

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[3] Ravi Vakil, *A Mathematical Mosaic,* Brendan Kelly Publishing, Inc., pp. 27-29. 1996
[4] Michael S. Schneider, *A Beginner's Guide to Constructing the Universe,* Harper Perennial, p 227, p 229-230. 1995
[5] Paul A. Calter, *Squaring the Circle,* Key College Publishing, pp. 147-148. 2008