# Teaching Group Theory Using Portraits of Groups

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## 1. Abstract

This paper looks at using the representation of small finite groups as groups of transformations of a compact surface of genus two or more to interest a general audience in the study of group theory. The ideas are presented in a very elementary way and the Portraits of Groups developed for previous Bridges conferences show interesting applications.

## 2. Introduction

The genesis of this paper is a talk that I gave to a non-mathematical audience in October 2007. The audience was primarily faculty members in the College of Liberal Arts at Towson University. Many of the faculty confided to me that they were afraid of mathematics and not very good at it. The response to the talk was very positive and everyone felt that they got something out of it. A PowerPoint presentation of the talk is available at [5].

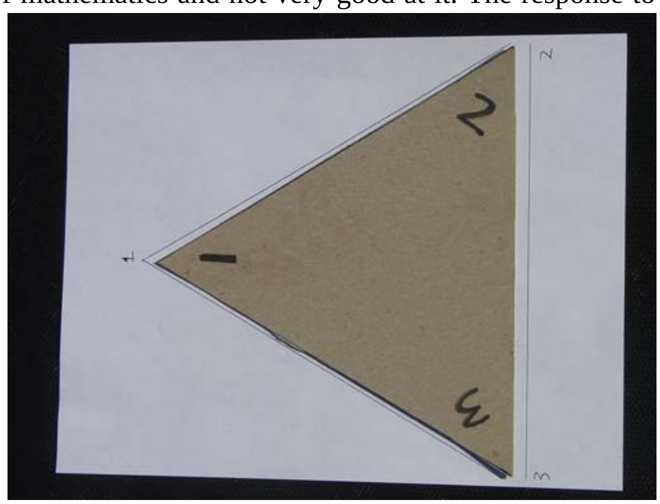


**Figure 1:** *Cardboard Equilateral Triangle*

The talk should begin with a description of what we are going to do. Specifically, a group is a mathematical object like a number system. This part was held to a minimum. The concept of symmetry was discussed next. The members of the audience were given a set of cardboard equilateral triangles and pieces of paper with a slightly larger equilateral triangle drawn on it (as shown in Figure 1). The question was posed, how many symmetries of this equilateral triangle are there? The most common answers to this question by college students are three, five, six and an infinite number. A symmetry is specified only by its starting and ending configuration. The paper and cardboard triangles make it easier to move the figure in the correct ways and it eliminates the answer infinity. This is done by saying that a symmetry or motion starts from the configuration shown. The triangle is picked up, moved and then set back down inside the outer lines again. It must also be made clear that the numbers on the triangle are not part of the figure, but are on the figure so that you may count the different motions that take the figure to itself. This also facilitates introducing the idea of composition of motions later in the talk. Note that the back side of the triangle has the same numbers on each corner as the front side, but they are green instead of black. Give the audience a few minutes to play with the triangles before asking how many motions they counted. Several answers are usually given. Encourage the audience to talk about how they got their count. Point out that "no motion" is a valid symmetry.  The audience arrives at a consensus of six motions.

List the motions. If the audience is more mathematically sophisticated you can bring in the idea of permutations of the corners. I named each of the six motions with a letter; I is no motion, R and S are the rotations and V, A and B are the reflections. The composition of motions is discussed next. Several examples are presented in detail, moving the triangle with first one motion and then another and identifying the result. Specifically, R times V is the same motion as B or R * V = B. After several minutes of this, let the audience do these calculations on their own. The partially complete "multiplication table" in Figure 2 is distributed.

|   | I | R | S | V | A | B |
|---|---|---|---|---|---|---|
| I | I | R | S | V | A | B |
| R | R | S | I |   |   |   |
| S | S | I | R |   |   |   |
| V | V |   |   |   |   |   |
| A | A |   |   |   |   |   |
| B | B |   |   |   |   |   |

**Figure 2:** *Multiplication Table*

The set of all motions with the operation of composition is an example of a group. This is the group of symmetries of the equilateral triangle and it has "order" six. This gives the audience a simple notion of a group. A "portrait" of a group is the division of a surface into equal regions that the group moves around. Each symmetry moves one surface region to another region. For the equilateral triangle these symmetries are rotations and reflections. The symmetry group of the equilateral triangle can be drawn on a sphere. One region of the surface is labeled the identity, and the other regions are labeled with the group element necessary to get from the identity region to that region. This gives a diagram as in Figure 3 and a model pictured in Figures 4 and 5. The models also show actions that preserve the orientation of the triangle as various light shades as in Figure 4 and actions that reverse the orientation as dark shades as in Figure 5. Since the letters on the two sides of the cardboard equilateral triangle are of different colors, it is fairly easy to explain orientation to any audience.

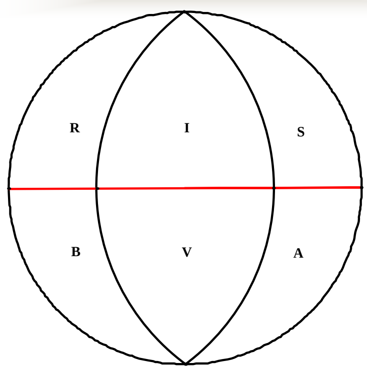


**Figure 3:** *Diagram of a Portrait*

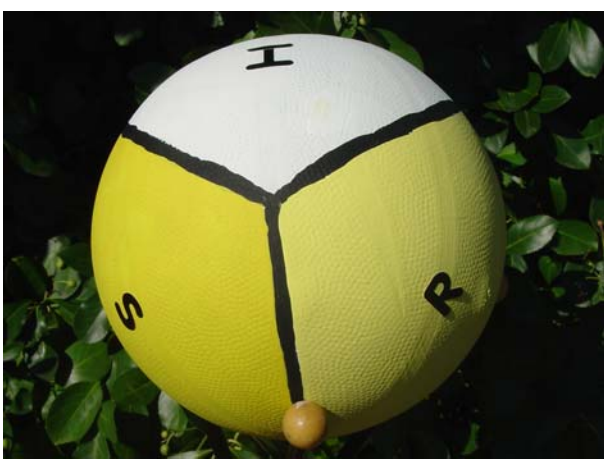


**Figure 4:** *Symmetries of the Triangle*


**Figure 5:** *Symmetries of the Triangle*

At this point in the talk, some mention of the history of groups and of these portraits is appropriate. I mention Burnsides' 1911 book and Figure 6 shows his portrait of the Quaternion Group, which has order 8. Figure 7 shows a portrait of the dicyclic group of order 12. Both groups act on a surface with two holes in it. I also point out that these portraits cannot be drawn on a surface with fewer holes in it. We say that these groups have strong symmetric genus 2. There are groups which have any desired genus. A group of order 16 which has three holes (genus 3) is also modeled in [3, page 134].

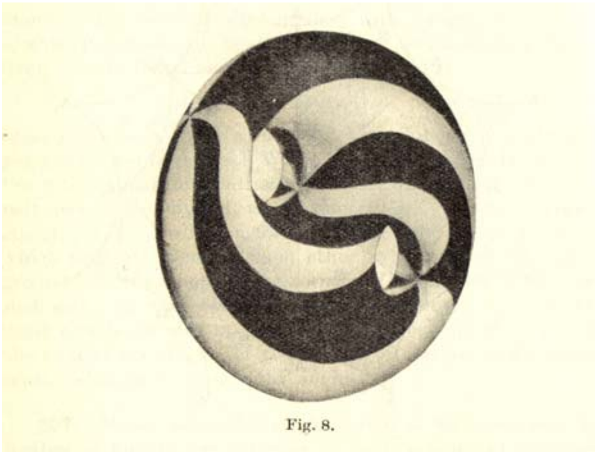


**Figure 6:** *Portrait of the Quaternion Group* [1, page 396]

**Figure 7:** *Portrait of the Dicyclic Group* [3, page 133]

At this point in the talk, the audience has some idea how groups are the "algebra of symmetries". It is time to show them some pretty pictures. One possibility is to look at the Euclidean plane groups and the corresponding Wallpaper patterns. This can be done in the context of real world examples or artistic examples. These correspond to infinite groups, but the surface in question is the Euclidean plane. For my talk in 2007, I opted for more complicated finite groups on a surface of genus 2.

Certain groups act on a surface as a quotient of a “hybrid triangle group”. This group must be treated differently because one of the generators “c” reverses orientation and the other generator “x” preserves orientation. The image of a region under the action of the generator “x” is another region of the same color. Regions of the same color cannot be adjacent and so the portraits of these groups must be represented somewhat differently. These groups are denoted by $HT(m,n)$ and the generators satisfy the relations $c^2 = x^m = [c,x]^n = 1$. Since “x” is a rotation by $360°/m$, some adjacent regions would be indistinguishable. Therefore, a fundamental region is divided into two regions with slightly different color so that the images under “x” can be distinguished. These differences are described in detail in the paper [4]. The group $HT(m,n)$ is infinite and it has a portrait on hyperbolic space.

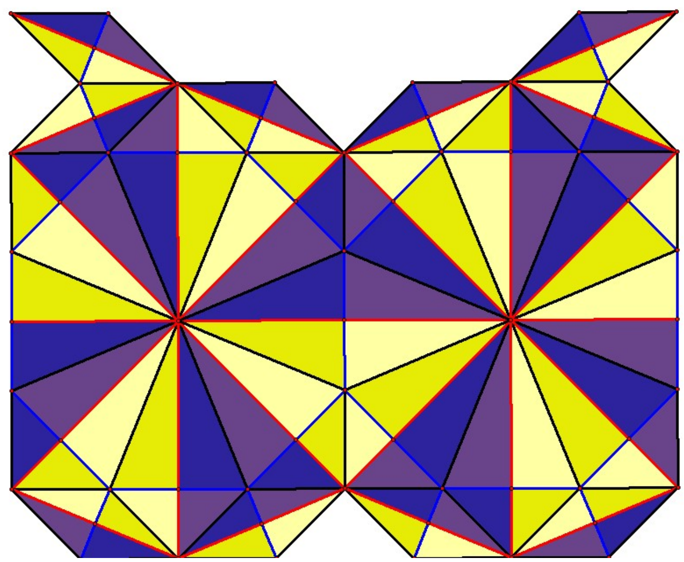

**Figure 8:** *Polygonal representation of $P_{48}$*

One such quotient of $HT(3,4)$ is the group, $P_{48}$, a group of order 48. We must start with a diagram like the diagram in Figure 3. The corresponding diagram for $P_{48}$ is given in Figure 8. Just like in Figure 3, when you exit through the edge on one side, you enter another region on the diagram. Therefore, the diagram in Figure 8 folds up into a colored surface. This surface also has two holes in it. A model of this surface is included in Figure 9. The surface would be the same as the surface in Figure 6, if all of the dark regions were the same color and all of the light regions were the same color. This reflects the fact that the quaternion group is also a quotient group of $P_{48}$.

The pictures in Figures 4, 5, 6, 7, 8 and 9 leave the audience with a sense that this material has interesting applications and the beginning hands-on activities give them a sense that they can understand it. The audience leaves with an appreciation for abstract mathematics that they may not have had before.

**Figure 9:** *A model of the Group* $P_{48}$ [4, page 114]

## 7. References


1. W. Burnside, *Theory of groups of finite order*, (Cambridge University Press 1911).
2. C. May and J. Zimmerman, Groups of small symmetric genus, Glasgow Math. J. 37(1995), p. 115 – 129.
3. J. Zimmerman, Portraits of Groups, Conference Proceedings 2006, Bridges: Mathematical Connections in Art, Music and Science, London, England, 131 - 134.
4. J. Zimmerman, Portraits of Groups II, Orientation Reversing Actions, Conference Proceedings 2007, Bridges Donostia: Mathematical Connections in Art, Music and Science, San Sebastian, Spain, 109 - 114.
5. J. Zimmerman, http://pages.towson.edu/zimmer/papers/Portraits2007.ppt.