Coxeter Groups in Colored Tilings and Patterns

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Abstract

This paper illustrates a number of ways that color symmetry theory can be used as a tool to study abstract groups such as Coxeter groups.

Introduction

People are always attracted to pictures and colors. Any presentation becomes more engaging for its intended audience if it involves colored figures that allow one to visualize certain concepts. A particularly interesting kind of illustration is a *tiling* or a *tessellation* which involves a countable collection of polygons, called *tiles*, that completely fill the plane without gaps or overlaps. Aside from polygons, one can also use motifs or symmetrical patterns to fill the entire plane, as seen in famous works by the Dutch artist M. C. Escher. These were shown by mathematicians such as H. S. M. Coxeter to possess certain group-theoretic properties.

Consider for instance the colored symmetrical patterns presented in Figure 1. The uncolored pattern of the Escher-inspired artwork *The Creatures* by Jos Leys [17] given in Figure 1(a) has symmetry group generated by a mirror reflection whose axis passes through the backbone of a creature, and a 3-fold rotation whose center does not lie on the axis of reflection. Figure 1(b) shows a black and white tiling on the upper half plane model in hyperbolic geometry representing the modular group $PSL(2, \mathbf{Z})$. A computer rendition of Escher's famous *Circle Limit IV* [12] appears in Figure 1(c) with color fixing symmetry group that includes a 4-fold rotation, and a 3-fold rotation with center lying on an axis of reflection.

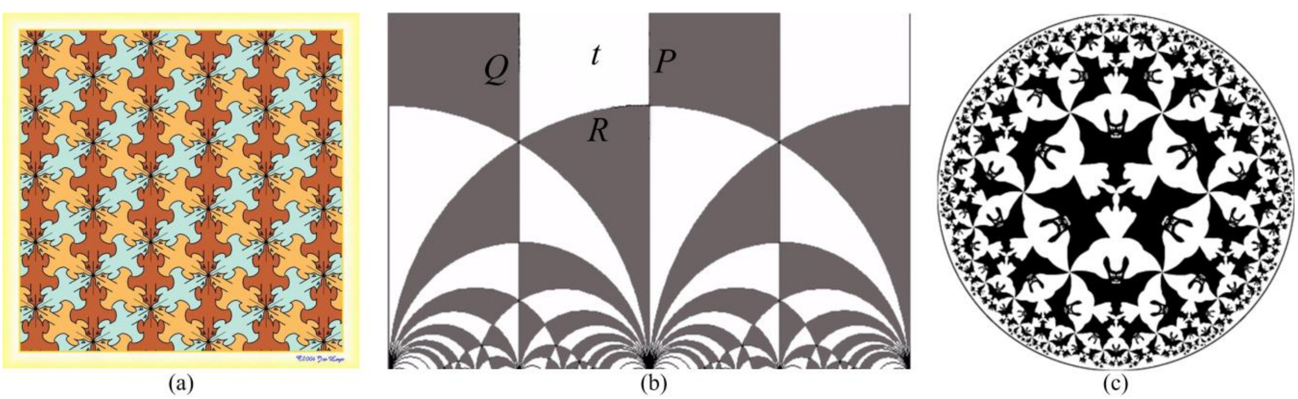


Figure 1. (a) *Jos Leys' Creatures 2;* (b) *A black and white tiling on the upper half plane by triangles with interior angles $\pi/3$, $\pi/2$ and 0;* (c) *a rendition of Escher's Circle Limit IV.*

Colored tilings and patterns are interesting tools in the study of abstract group theory. Through color symmetry, one can study symmetry groups and their subgroups and cosets, as well obtain insight into conjugacy, group extensions and other algebraic concepts. In this paper, we discuss our approach to the study of abstract groups using colored symmetrical tilings and patterns. In particular, we focus our discussion on certain examples of Coxeter groups.

Representing Certain Coxeter Groups Geometrically

A group generated by n reflections S_1, S_2, …, S_n satisfying the relation $(S_iS_j)^{p_{ij}} = 1$, $1 \le i \le j \le n$ where $p_{ii} = 1$ and $p_{ij} \ge 2$ for $i \ne j$, is a *Coxeter group*.

An example of a Coxeter group is the *triangle group* *pqr generated by three reflections P, Q, and R with the following presentation

$$G = \langle P, Q, R \mid P^2 = Q^2 = R^2 = (PQ)^r = (QR)^p = (PR)^q = 1\rangle.$$

In the two-dimensional plane, the group *pqr is manifested geometrically by considering a fundamental triangle t where P, Q, and R respectively denote reflections in the sides of the triangle opposite the interior angles π/p, π/q, and π/r (p, q, r integers ≥ 2). The triangle t is on the spherical, Euclidean, or hyperbolic planes according as $1/p + 1/q + 1/r$ is greater than, equal to, or less than 1, respectively. Repeatedly reflecting the triangle t in its sides results in a tiling of the appropriate plane by copies of the triangle. If p, q, r are distinct then G is the symmetry group of the tiling. Otherwise, G is a subgroup of the symmetry group of the tiling.

The groups *332, *432 and *532 for example, give rise to tilings of the sphere. On the other hand, the groups *442, *632 and *333 correspond to tilings on the Euclidean plane.

In the hyperbolic plane, there is an immense variety of triangle groups. An example is the symmetry group of the tiling by triangles with interior angles $\pi/6$, $\pi/4$ and $\pi/2$ shown in Figure 2(a) which is the hyperbolic triangle group *642. A special case of the hyperbolic triangle group *pqr occurs when the product of the two generators, say P and Q, for instance, have infinite order. The axes of these reflections determine a zero angle, giving rise to a vertex at infinity. In this situation, the triangle group is called the *generalized extended Hecke group*, denoted by *$pq\infty$. A well-known group belonging to this class is the extended modular group *32∞. Its index 2 subgroup, the modular group $32\infty \cong PSL(2, \mathbf{Z})$, has been of great interest in number theory and automorphic theory. The tiling (disregarding color black) given in Figure 1(b) has symmetry group *32∞. Another example is the extended Hecke group *42∞. The tiling on the upper half plane by triangles with interior angles $\pi/4$, $\pi/2$ and 0 shown in Figure 2(b) has symmetry group *42∞.

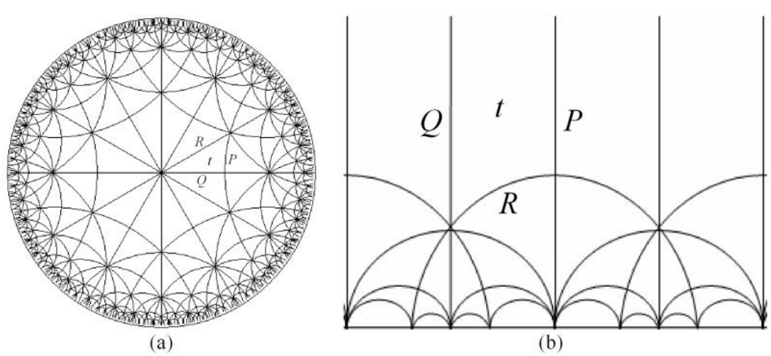
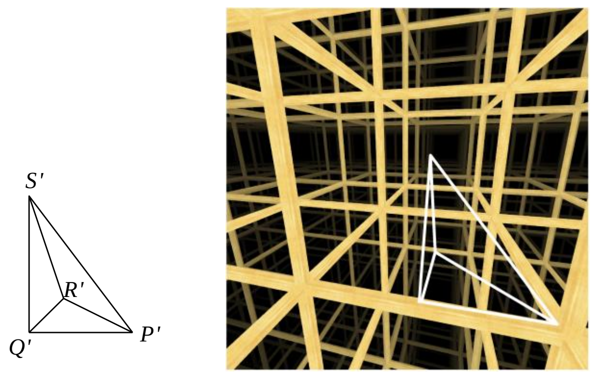


Figure 2. *A tiling by triangles on the hyperbolic plane with symmetry group* (a) *642*;* (b) *42∞.

A Coxeter group generated by four reflections P, Q, R, S with the following presentation

$$G' = \langle P, Q, R, S \mid P^2 = Q^2 = R^2 = S^2 = (PQ)^p = (QR)^q = (RS)^r = (PR)^2 = (PS)^2 = (QS)^2 = 1\rangle$$

is the symmetry group $[p, q, r]$ of a regular honeycomb $\{p, q, r\}$ in spherical, Euclidean or hyperbolic 3-space. A *regular honeycomb* $\{p, q, r\}$ ($1/p + 1/q > 1/2$ and $1/q + 1/r > 1/2$) is a collection of nonoverlapping Platonic solids {*p*, *q*} (faces are *p*-gons {*p*} meeting *q* at a vertex) such that each face {*p*} belongs to just two of them, and every edge to *r* of them. The notation {*p*, *q*, *r*} indicates that each *cell* is a platonic solid {*p*, *q*} and that *r* such cells come together at an edge. The possible regular honeycombs are {3, 3, 3}, {4, 3, 3}, {3, 4, 3}, {5, 3, 3}, {4, 3, 4}, {5, 3, 4}, {3, 5, 3} and {5, 3, 5}. The first four are spherical, the next is Euclidean and the last three are hyperbolic. Each regular honeycomb {*p*, *q*, *r*} has a fundamental tetrahedron *P'Q'R'S'* having dihedral angles π/p, π/q, π/r, $\pi/2$, $\pi/2$ and $\pi/2$. The respective dihedral angles at the edges *R'S'*, *P'S'*, *P'Q'* are π/p, π/q, π/r. The reflections *P*, *Q*, *R*, *S* have mirror planes the faces opposite the vertices *P'*, *Q'*, *R'*, *S'*, respectively (see Figure 3(a)). Repeatedly reflecting *P'Q'R'S'* along its faces results in a regular honeycomb of the appropriate space. Figure 3(b) shows the regular {4, 3, 4} honeycomb in Euclidean space with symmetry group [4, 3, 4]. A fundamental tetrahedron is outlined in white.



(a) (b)

Figure 3. (a) *A fundamental tetrahedron P'Q'R'S'* ; (b) *The regular* {4, 3, 4} *honeycomb.*

Perfect Colorings of Tilings

In our work, a problem we are addressing is the determination of the subgroup structure of hyperbolic Coxeter groups. Considering the geometric representations of some of these groups in the plane and in space, our approach to the problem will be to study perfect colorings of the corresponding tilings and honeycombs, respectively. A *perfect coloring* of a given tiling is a coloring where all the symmetries of the uncolored tiling map all parts of the tiling having the same color onto parts having a single color, that is, the symmetries permute the colors. In this case, we also say the tiling is *perfectly colored*.

In [10] for example, a method was provided to determine index 2 subgroups of triangle groups. Given a triangle group *G* which is the symmetry group of an uncolored tiling by triangles in a plane, the approach employed to determine index 2 subgroups of *G* is to construct black and white colorings of the tiling which are perfectly colored. The method is based on the following result that appears in [10].

Theorem 1. Let *t* be a tile with trivial symmetry group and *T* a tiling of the plane by copies of *t* which fill the plane with no gaps and overlaps. Let *G* be the symmetry group of the tiling and assume that *G* acts transitively on the tiling. Then

(*i*) In a perfect black and white coloring of *T* (where tile *t* is colored white), the elements of *G* which fix the colors form a subgroup *K* of *G* of index 2.

(*ii*) If *K* is a subgroup of *G* of index 2 and the tiles in the *K*-orbit of *t*, $Kt = \{kt: k \in K\}$ are colored white and the rest of the tiles are colored black, then the resulting coloring is a perfect coloring.

(*iii*) The map that assigns to each perfect black and white coloring of *T* (where *t* is colored white) the subgroup of elements of *G* which fix the colors is a one-to-one correspondence between the set of perfect black and white colorings of *T* (with *t* colored white) and the set of subgroups of *G* of index 2.

The one-to-one correspondence results from the fact that in a perfect black and white coloring of T with t colored white, the subgroup K consists of the elements which fix the colors. Hence the tiles colored white are precisely the elements in the K-orbit Kt and the rest of the tiles (the tiles in aKt) must be the tiles colored black.

To illustrate the method given above, we derive the index 2 subgroups of the extended Hecke group $G = \langle P, Q, R\rangle \cong *p2\infty$, $p > 2$. To obtain all index 2 subgroups of G, we consider perfect black (B) and white (W) colorings of T where all elements of G effect permutations of black and white. The triangle tiling T we consider here are copies of a fundamental triangle t with interior angles π/p, $\pi/2$, and the 0 angle, where $Stab_G(t) = \{e\}$. For a perfect black and white coloring of T, a homomorphism π: $G \to \{I, (B\ W)\}$ is defined. Since $G = \langle P, Q, R\rangle$, π is completely determined when $\pi(P)$, $\pi(Q)$ and $\pi(R)$ are specified. To construct the coloring, a fundamental triangle t is considered and assigned color white. If G is to permute the colors, then each of its generators P, Q, R either fixes the colors or interchanges them. There are a total of 7 possibilities; the case where each of P, Q and R fixes the colors is excluded. The possible perfect colorings, where each of P, Q, R either fixes (0) or interchanges (1) the colors is given in Table 1.

Observe that a rotation about a vertex of T either fixes the colors or interchanges black and white. If the given rotation is of odd order n, and the rotation interchanges black and white, then the nth power of this rotation will send a triangle colored black to a triangle colored white. This is contrary to the fact that the nth power of the rotation is the identity transformation which fixes the colors. Thus the situations where a rotation about a vertex interchanges the colors are applicable only when the given rotation is of even order. On the other hand the situation where the rotation fixes the colors always occurs regardless of the order of the given rotation. If the rotations at every vertex of a triangle in T are of even order, then all seven colorings given in Table 1 arise. However, if a rotation of odd order exists at exactly one vertex of a triangle, then there are only three colorings, those corresponding to 1, 2 and 5 in Table 1. Example, for the group $G = \langle P, Q, R\rangle \cong *p2\infty$, if QR is of odd order, then the only index 2 subgroups of G are $p2\infty$, $*pp\infty$ and $p*\infty$.

	Generators of $*p2\infty$ P	Q	R	Generators for the subgroup fixing the colors	Subgroup fixing the colors (in Conway notation)
1	1	1	1	QP, RP	$p2\infty$
2	1	0	0	R, Q, PQP	$*pp\infty$
3	0	1	0	P, R, QPQ, QRQ	$*(p/2)22\infty$
4	0	0	1	P, Q, RQR	$*(p/2)\infty\infty$
5	0	1	1	P, RQ	$p*\infty$
6	1	0	1	Q, RP, PQP	$2*(p/2)\infty$
7	1	1	0	R, QP	$\infty*(p/2)$

Table 1. *Colorings arising from the cases where each of P, Q, R either fixes* (0) *or interchanges* (1) *the colors.*

Let us look at some examples. A tiling by a fundamental triangle t with interior angles $\pi/4$, $\pi/2$ and 0 will give 7 perfect black and white colorings (Figure 4(a) – (g)) corresponding to the 7 index 2 subgroups of $*42\infty$, which are 42∞, $*44\infty$, $*222\infty$, $*2\infty\infty$, $4*\infty$, $2*2\infty$ and $\infty*2$ respectively. On the other hand, if we consider t with interior angles $\pi/3$, $\pi/2$ and 0, there will be three perfect black and white colorings of T (Figure 4(h) – (j)). Consequently, this will give respectively, three index 2 subgroups of $*32\infty$: 32∞, $*33\infty$ and $3*\infty$.

Now, it is also possible to determine the index 2 subgroups of $G' = [p, q, r]$, the symmetry group of the regular honeycomb $\{p, q, r\}$, by considering perfect black and white colorings of the honeycomb. In this case, a fundamental tetrahedron will be given color white. If G' is to permute the colors, then each of its generators P, Q, R, S either fixes the colors or interchanges them. A total of fifteen colorings will result if each of p, q and r are even. The list is given in Table 2. Considering specific examples we obtain: corresponding to the {4, 3, 4} Euclidean honeycomb there will be seven index 2 subgroups (subgroup nos 1, 2, 5, 8, 9, 12, 15 in the list). The hyperbolic group [5, 3, 4] has three index 2 subgroups

(subgroups nos 1, 5, 12); the hyperbolic groups [3, 5, 3] and [5, 3, 5] have one index 2 subgroup each (subgroup no 1).

This approach of using perfect black and white colorings to determine index 2 subgroups may be applied to determine subgroups of index larger than 2 of groups *pqr and [p, q, r], and is work in progress.

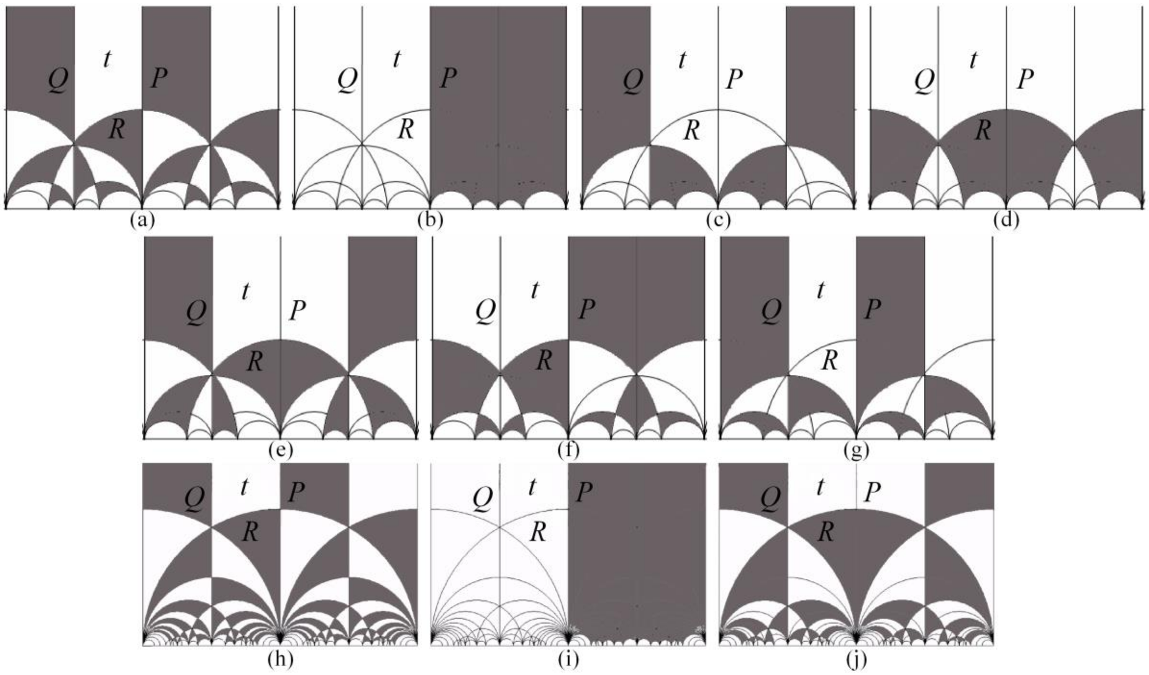


Figure 4. *The black and white colorings of the tiling by triangles with interior angles* (a) – (g): $\pi/4$, $\pi/2$ *and* 0; (h) – (j): $\pi/3$, $\pi/2$ *and* 0

	Generators of [p, q, r] P	Q	R	S	Generators for the subgroup fixing the colors		Generators of [p, q, r] P	Q	R	S	Generators for the subgroup fixing the colors
1	1	1	1	1	QP, RP, SP	9	0	1	1	0	P, RQ, S
2	1	0	0	0	Q, R, S, PQP	10	0	1	0	1	P, R, SQ, QRQ
3	0	1	0	0	P, R, S, QPQ, QRQ	11	0	0	1	1	P, Q, SR
4	0	0	1	0	P, Q, S, RQR, RSR	12	1	1	1	0	QP, RP, S
5	0	0	0	1	P, Q, R, SRS	13	1	1	0	1	QP, R, SP
6	1	1	0	0	QP, R, S	14	1	0	1	1	Q, RP, SP
7	1	0	1	0	Q, RP, S, PQP	15	0	1	1	1	P, RQ, SQ
8	1	0	0	1	Q, R, SP						

Table 2. *Colorings arising from the cases where each of P, Q, R, S either fixes (0) or interchanges (1) the colors.*

A specific type of perfect coloring of a tiling where every color appears once in each vertex of the tiles is called a *precise perfect coloring* of the tiling. Such colorings are possible avenues to discover interesting algebraic properties. Consider for instance precise perfect colorings of tilings associated with triangle groups generated by three reflections having presentation

$$*n32 = \langle P, Q, R \mid P^2 = Q^2 = R^2 = (QR)^n = (PR)^3 = (PS)^2 = 1\rangle,\ n \geq 7.$$

The triangle group *n32, $n \geq 7$ is the symmetry group of the regular 3^n tiling on the hyperbolic plane. A systematic way of constructing precise perfect colorings of a 3^n tiling using n colors is given in [15]. See Figure 5 for examples of precise perfect colorings of the 3^7, 3^8 and 3^9 tilings. It is interesting to note that the simple group of order 168 is the group of permutations of the precise perfect coloring of the 3^7 tiling given in Figure 5(a). In [18], this 7-coloring was shown to establish the isomorphism between $PSL(2, \mathbf{Z}_7)$

and $SL(3, \mathbf{Z}_2)$. It would be worth investigating whether other interesting group-theoretic consequences also follow from precise perfect colorings of other 3^n tilings.

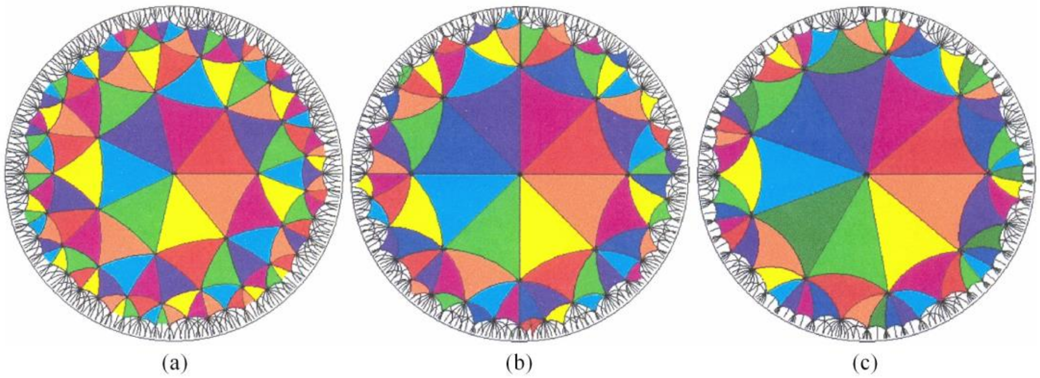


Figure 5. *A precise perfect coloring of the* (a) 3^7 *tiling;* (b) 3^8 *tiling;* (c) 3^9 *tiling.*

Non-Perfect Colorings of Tilings

A coloring of a given tiling where not all symmetries of the uncolored tiling permute the colors is called *non-perfect*. Non-perfectly colored patterns and designs occur more frequently in art than perfectly colored ones, and are good illustrations to highlight certain isometries over the others. Figure 6(a) shows Barbara Pickett's *Monreale* [19], a handwoven silk velvet on a Jacquard loom. The symmetry group of the uncolored pattern is of type *2222, a right angled Coxeter group, generated by reflections in the four sides of a rectangle. The subgroup that permutes the colors is an index 4 subgroup of Conway type o. The effect is to highlight the translational elements of the group *2222. Another example is the *Hyperbolic Spiderweb* shown in Figure 6(b), Tony Bomford's first hyperbolic rug [11]. This rug was inspired by M.C. Escher's *Circle Limit IV*. The *Hyperbolic Spiderweb* displays non-perfect color symmetry. The reflection whose axis is a horizontal line passing through the center of the central 6-gon does not permute the colors. There are three shades of tan (the lightest almost white) in Bomford's rug. The horizontal reflection interchanges tan1 and tan3 in the eighth (outer) hexagonal ring, but interchanges tan2and tan3 in the sixth hexagonal ring.

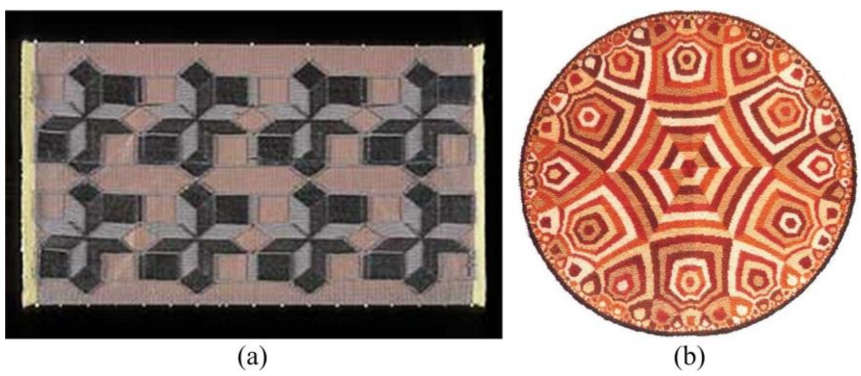


Figure 6. (a) *Monreale by Barbara Pickett;* (b) *A non-perfect coloring of* $*32\infty$.

The colorings of the tiling given in Figure 2(a) and presented in Figures 7(a) – (e) are non-perfect. These colorings have been constructed using right cosets of a subgroup S of G, the symmetry group of the given tiling. If S is a subgroup of G of index n, by a *coloring using the right cosets* of S we refer to a bijective map from the set of right cosets of S to a set of n colors. Tiles labeled by the elements of a right coset are colored using the color assigned to the right coset.

Non-perfect colorings of a given tiling, just like the perfect colorings, can help characterize the subgroup structure of the symmetry group G of the tiling. For example, non-perfect colorings using right cosets of a subgroup S of G can provide information regarding conjugate subgroups. The following theorem [15] is helpful in the study of the properties of S and its conjugates.

Theorem 2. Let S be a subgroup of a group G and $N := N_G(S) = \{g \in G : gS = Sg\}$, the normalizer of S in G. Let C be the set of right cosets $\{Sa : a \in G\}$. Then N acts on C by left multiplication. Under this action, two right cosets Sa and Sb (a, $b \in G$) are in the same N-orbit if and only if $a^{-1}Sa = b^{-1}Sb$. Moreover, the number of distinct conjugates of S in G is $[G : N]$.

The preceding theorem is true since if $Sb = Sa$ are in the same N-orbit, then there exists $n \in N$ such that $Sb = nSa = Sna$ or $nab^{-1} \in S \leq N$. Therefore, $ab^{-1} \in N$ and $ab^{-1}S = Sab^{-1}$ or $a^{-1}Sa = b^{-1}Sb$. Conversely, $a^{-1}Sa = b^{-1}Sb$ implies $Sba^{-1} = ba^{-1}S$ which means that $ba^{-1} \in N$ and $Sb = ba^{-1}(Sa)$. That is, Sb and Sa are in the same N-orbit. What this means is that if two colored regions corresponding to the right cosets Sa and Sb of a subgroup S are of the same shape and we can get one by applying an element of the subgroup N on the other, then $a^{-1}Sa$ and $b^{-1}Sb$ belong to the same conjugacy class. Otherwise, $a^{-1}Sa$ and $b^{-1}Sb$ are distinct conjugates. Using the orbit-stabilizer theorem, the number of distinct conjugates can be shown to be precisely the number of distinct N-orbits, $[G : N]$.

To illustrate the above ideas, let us look at the hyperbolic triangle group $G = \langle P, Q, R\rangle \cong *642$. Consider for example, a right coset coloring using a non-normal index 3 subgroup S_1 of G given in Figure 7(a). It can be observed that the coloring has three colors corresponding to the three cosets of S_1. Note that the cosets given colors white and dark gray make up two different quadrilaterals, respectively; a quadrilateral consisting of 4 triangles and a quadrilateral consisting of 8 triangles. The cosets given color light gray consist of triangles. The three cosets assume different shapes, and so we cannot send one coset to another by an element of $N_G(S_1)$. Thus, each color would belong to three different $N_G(S_1)$-orbits. Hence, there are three distinct conjugate subgroups. The right coset colorings corresponding to the conjugate subgroups of S_1 appear in Figure 7(a) – (c).

Figure 7(d) shows a right coset coloring using an index 4 subgroup S_2 of G with four colors. The cosets colored white and black consist of octagons while the cosets colored light and dark gray consist of triangles. It follows that the white and black cosets belong to the same conjugacy class while the light and dark gray cosets belong to another. Thus there is only one subgroup conjugate to S_2, whose right coset coloring is shown in Figure 7(e).

Now, an important property of colorings using right cosets is the following: a coloring using right cosets of a subgroup S and a coloring using right cosets of a conjugate subgroup are equivalent in the sense that except for the choice of colors used, one may be obtained from the other by a given symmetry of the uncolored pattern. This is so since if $a \in G$, $a\{Sg : g \in G\} = \{aSg : g \in G\} = \{(aSa^{-1})ag : g \in G\}$. This means that a takes the right cosets of S to the right cosets of aSa^{-1}.

Consider again the right coset coloring given earlier in Figure 7(a). The coloring is obtained using the subgroup S_1 generated by Q, P, RPR, $RQRQR$. Applying the reflection R to this coloring will give the right coset coloring shown in Figure 7(b). Thus the coloring in Figure 7(b) is a coloring using right cosets of S_1 conjugated by R. More particularly, this is a coloring using right cosets of S_1' where $S_1' = \langle RQR, RPR, P, QRQ\rangle = RS_1R^{-1} = R\langle Q, P, RPR, RQRQR\rangle R^{-1}$. Similarly, applying the rotation QR to the given coloring in Figure 7(a) will yield the right coset coloring presented in Figure 7(c). The coloring in Figure 7(c) is a coloring using right cosets of S_1 conjugated by QR. Moreover, this is a coloring using cosets of S_1'', where $S_1'' = \langle QRQRQ, QRPRQ, QPQ, R\rangle = (QR)S_1(QR)^{-1} = (QR)\langle Q, P, RPR, RQRQR\rangle(QR)^{-1}$.

Conclusion and Outlook

In this paper, we have shown that perfect and non-perfectly colored symmetrical patterns can be used as tools to understand and visualize structural properties of abstract groups. For instance, we have illustrated how perfect colorings of tilings facilitate the derivation of index 2 subgroups of associated hyperbolic Coxeter groups, particularly those generated by three or four reflections. As a next step, a similar approach may be employed to derive higher index subgroups of these groups as well as other types of Coxeter groups. It would also be interesting to study more closely non-perfect colorings using right cosets to determine information on normal and conjugate subgroups. This paper highlights the contribution of color symmetry theory to algebra, and further interconnections between color symmetry and algebra may be explored in future work.

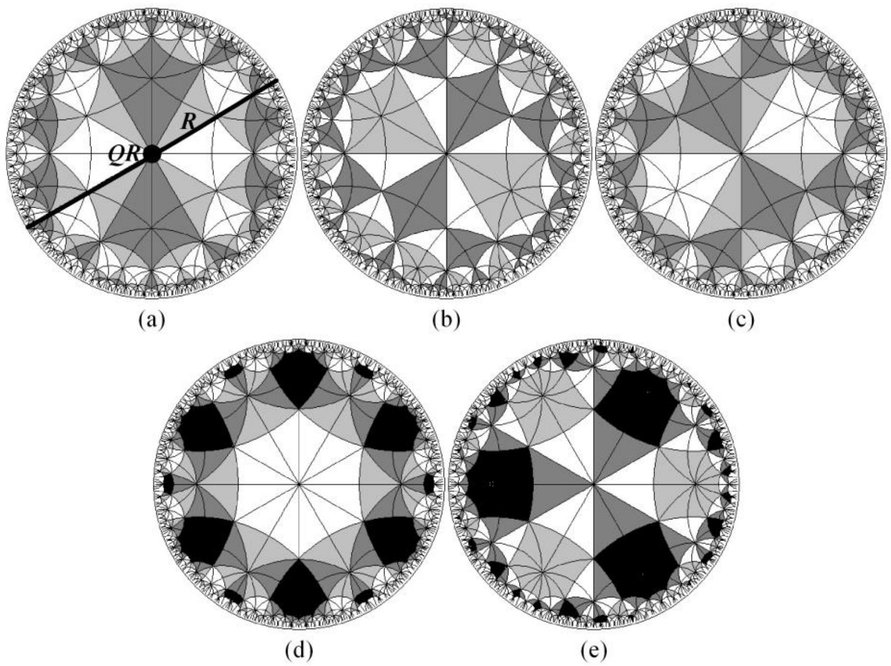


Figure 7. *Right coset coloring/s using* (a)-(c) *Conjugate subgroups of* S_1; (d)-(e) *Conjugate subgroups of* S_2.

References

[1] Coxeter, H.S.M. *Coloured Symmetry*, in M.C. Escher Art and Science, The Netherlands, H.S.M. Coxeter, et al., Eds, 15 – 33 (Elsevier Science Publishers B.V., The Netherlands, 1985).
[2] Coxeter, H.S.M. *Discrete Groups Generated by Reflections*, in The Annals of Mathematics, 2nd Ser. Vol. 35, No. 3, 588 – 621 (1934).
[3] Coxeter, H.S.M. *Twisted Honeycombs*, Regional Conference Series in Mathematics No. 4. USA: American Mathematical Society (1970).
[4] Coxeter, H.S.M. and W.O. Moser. *Generators and Relations for Discrete Groups*, 2nd ed. USA: Springer-Verlag (1965).
[5] *Cubic Honeycomb*. http://en.wikipedia.org/wiki/Image:Cubic_honeycomb.png.
[6] Decena, M.C.B. *On The Index 3 and 4 Subgroups of Triangle Groups*, a PhD Dissertation, Ateneo de Manila University (2007).
[7] De Las Peñas, M.L.A.N., R. Felix and G. Laigo, *Colorings of Hyperbolic Plane Crystallographic Patterns*, in Z. Kristallogr 221, 665 – 672 (2006).
[8] De Las Peñas, M.L.A.N., G. Laigo and E. Provido, *Studying Non-Euclidean Tilings and their Groups with the Aid of Technology*, in Proceedings of the Eleventh Asian Technology Conference in Mathematics ATCM 2006, Hong Kong, W.C. Yang, et al., Eds, 317 – 326 (ATCM, Inc., USA, 2006)
[9] De Las Peñas, M.L.A.N., G. Laigo and E. Provido, *Hyperbolic Semi-Regular Tilings and their Symmetry Properties*, in Bridges Donostia: Mathematical Connections in Art, Music and Science Proceedings 2007, R. Sarhangi and J. Barallo, eds. UK: Tarquin Publications (2007).
[10] De Las Peñas, M.L.A.N., R. Felix and E. Provido, *On Index 2 Subgroups of Hyperbolic Symmetry Groups*, in Z. Kristallogr 222, 443 – 448 (2007).
[11] Dunham, D, *H.S.M. Coxeter and Tony Bomford's Colored Hyperbolic Rugs*, in Renaissance Banff, Bridges: Mathematical Connections in Art, Music, and Science Proceedings 2005, R. Sarhangi, ed. USA: Central Plain Book Manufacturing (2005).
[12] _____, *Transformation of Hyperbolic Escher Patterns*, in Visual Mathematics, Volume 1, No 1, (1999). http://www.d.umn.edu/~ddunham/isis4/index.html.
[13] Magnus, W. *Non-Euclidean Tesselations and their Groups.* USA: Academic Press Inc (1974).
[14] Grove, L.C. and C.T. Benson. *Finite Reflection Groups*, 2nd ed, Graduate Texts in Mathematics 99, USA: Springer-Verlag (1985).
[15] Hernandez, N.H.S. *On Colorings Induced by Low Index Subgroups of Some Hyperbolic Triangle Groups*, a Masters Thesis, University of the Philippines – Diliman (2003).
[16] Humphreys, J.E. *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics 29. UK: Cambridge University Press (1990).
[17] Leys, J. *The Creatures*. (2006). http://www.josleys.com/show_image.php?image=Escher/escher031.jpg&name=Creatures2.
[18] Mackenzie, D. *A Hyperbolic Plane Coloring and the Simple Group of Order 168*, in The American Mathematical Monthly, Vol 102, No 8 706 – 715 (1995).
[19] Pickett, B.S. *Monreale*, http://art-uo.uoregon.edu/faculty/gallery/show.cfm?image=29.
[20] Schwarzenberger, R.L.E. *Colour Symmetry*, in Bulletin of the London Mathematical Society, 16, 209 – 240 (1984).
[21] Senechal, M. *Color Symmetry*, in Computers & Mathematics with Applications, 16, 5 – 8, 545 – 553 (1988).
[22] Van Der Waerden, B.L. and J.J. Burckhardt, *Farbgruppen*, in Z. Kristallogr 115, 231 – 234 (1961).