Making Patterns on the Surfaces of Swing-Hinged Dissections

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Abstract

The article presents some examples regarding the illustrations of patterns and designs on the surfaces of some swing-hinged dissections. These patterns are made in a way that when they are swung from one shape of the dissection to another, the patterns are also changed along with the shapes.

1. Introduction

During the 2007 Bridges Conference in Spain, *Bridges Donostia*, Greg N. Frederickson of the Department of Computer Science, Purdue University, presented a paper, titled *Symmetry and Structure in Twist-Hinged Dissections of Polygonal Rings and Polygonal Anti-Rings* [10]. Frederickson showed a number of animations of different dissections including swing-, piano-, and twist-hinged dissections. His recent presentation, and another one during the 2005 Bridges Conference in Banff, Canada [9], motivated me to study this topic.

By a dissection we mean, cutting a geometric, two-dimensional shape into pieces that one can rearrange to form a different shape. Examples of interesting and mathematically sophisticated dissections come from a wide range of resources, from the ancient Greeks [1], to the medieval period of Islamic art and science [2, 3, 17], to the mathematical puzzle columns in magazines, and to many website resources [5, 6, 14, 15, 20].

A special property for some dissections is the ability to connect pieces by hinges in such a way that it preserves the transformation between two shapes by swinging one figure to another. Henry Dudeney demonstrated such a hinged dissection of an equilateral triangle to a square about a century ago [4]. Since then, a large number of hinged dissections have been discovered, which has resulted in an exciting book titled *Hinged Dissections: Swinging and Twisting* [8]. Frederickson has written another book about dissections that, mathematically speaking, along with the previously mentioned one, are necessary and sufficient conditions for a thorough understanding of the subject matter [7].

As an example, we look at Henry Dudeney's triangle-to-square dissection (Figure 1).

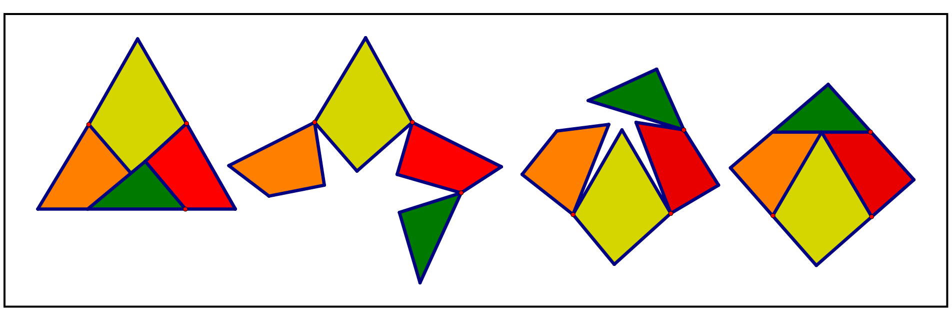


Figure 1: *From left to right, transformation of an equilateral triangle into a square.*

The purpose of this paper is to inspect the dissections' structures from a different point of view: Using the transferable surfaces to illustrate patterns and designs that are changeable in a meaningful and pleasing way along with the shapes of the dissections.

There are examples of changeable patterns using dissections. Slavik Jablan has illustrated spirals that can be changed into labyrinths that use dissections and rotations [11].

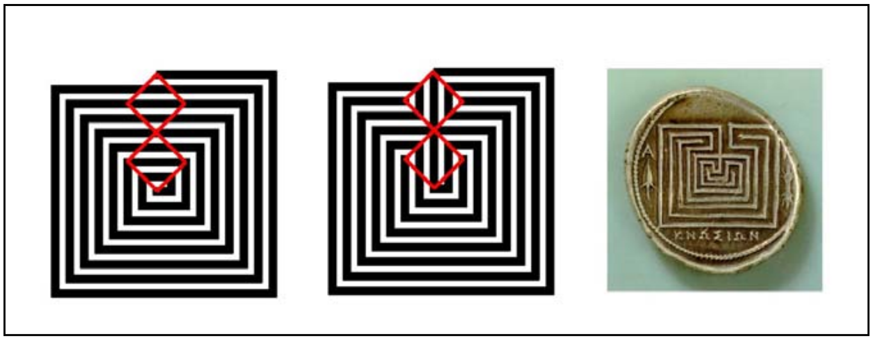


Figure 2: *Jablan shows that a simple dissection and rotation of 90° changes a simple spiral into a sophisticated labyrinth. Wouldn't it be possible that this discovery was so exciting to Greeks that they put it on their coins?* [21]

The above example demonstrates that in some cases, the change in patterns on the surface becomes more interesting than the change on the boundaries of the shapes. In fact, if we perform such a dissection-rotation to a blank square, there is nothing to amuse us.

We have already enjoyed Henry Dudeney's triangle-to-square dissection and feel nothing can make this puzzle more interesting. However, if we change our point of view, from mathematics into art, and use the surfaces, rather boundaries, to demonstrate changeable shapes, we may be able to contribute to the subject of dissection from a totally different perspective. The following figure shows an example of this idea.

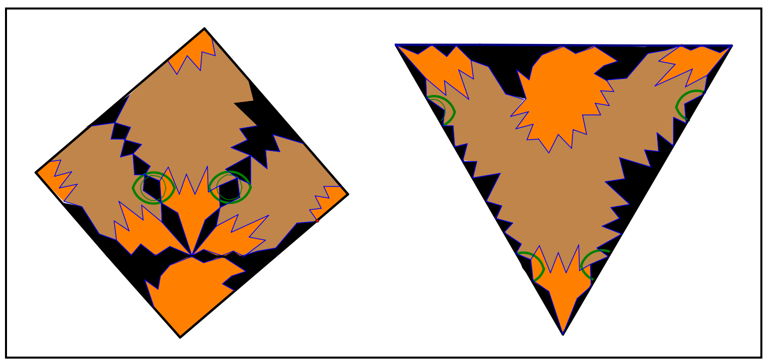


Figure 3: *The Transformation of a beast to an eagle using Dudeney's triangle-to-square dissection.*

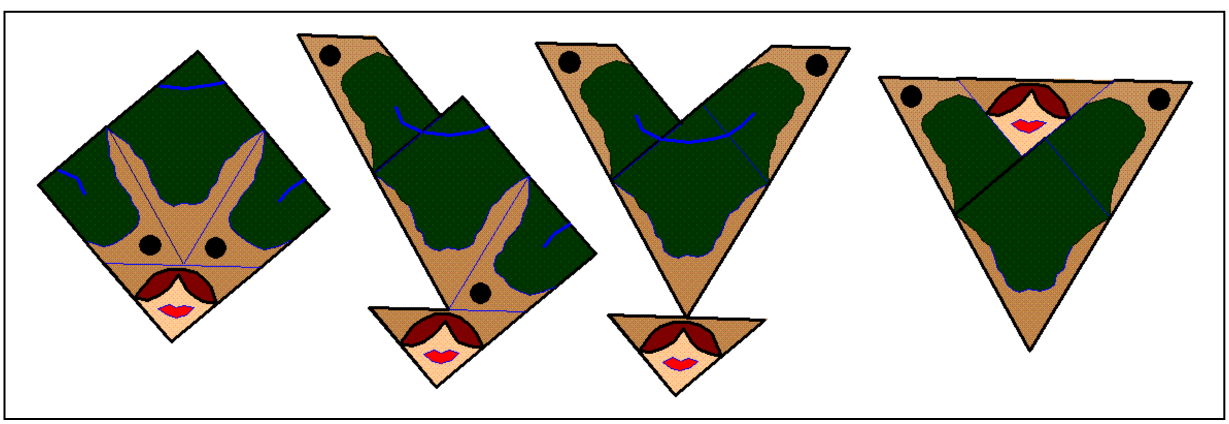


Figure 4: *The Transformation of a rabbit to a football player shows us how we have used two half-turn rotations and a translation to swing from one shape to another.*

We note that in general we can not claim any mathematical approaches to find a pair of suitable and pleasingly changeable patterns and designs for these surfaces. However, in our path of finding such patterns, we may use some mathematical properties to short cut the process. For example, if we study Figure 1 carefully we notice that, mathematically speaking, from the four pieces, the two side pieces use rotations of 180° to swing but the triangle piece on the bottom uses a simple translation to complete the process. The quadrilateral in the middle does not move. Figure 4 shows how these transformations change a rabbit to a football player.

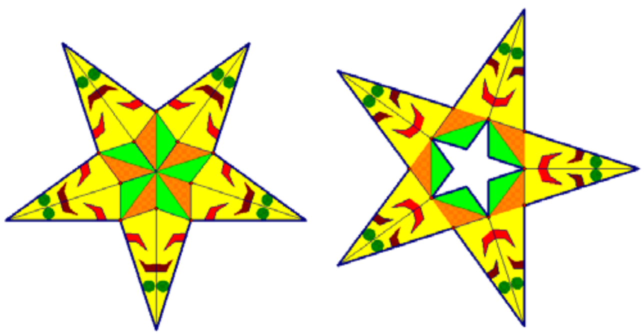


Figure 5

Figure 5 exhibits the transformation of a pentagram to a punched pentagram. The patterns that are added not only transform a sad pentagram to a happy one, but also transform a five-pointed star to a pentagram with a pentagram hole. This hinged game consists of five congruent triangles and five congruent quadrilaterals. It is the Varsady's Pentagrams for y/x = tan (2π/5) [8]. We use a rotation of 2π/5 degrees to transform one pentagram to another.

2. An Old Dissection

Among numerous mathematics manuscripts, Abûl-Wefâ Buzjani (940-998), wrote a treatise: *On Those Parts of Geometry Needed by Craftsmen* [3]. An interesting cutting-and-pasting problem that Abûl-Wefâ demonstrates in the treatise is the composition of a single square from a finite number of squares with different sizes. He first solves the problem for two squares and then comments that the solution can be extended for any number of squares (Figure 6). His solution is as follows (See Figure 7): We first put the small square (*a*) on the top of the larger square (*b*) as presented in (*c*) and then draw necessary line segments presented in (*d*) and (*e*) and finally cut the solid lines and paste them to obtain the resulting square [19].

For the mathematics justification, let a be the length of a side of a small square and b the length of a side of the large square. Then the sum of the areas of these two squares will be a^2+b^2. Now the cuts will create four right triangles with sides a and b (and hypotenuse $\sqrt{a^2+b^2}$), and a square with a side equal to $b-a$. The way that we arrange these four right triangles and the little square is in fact the visualization of the following equation: $a^2+b^2 = 4(1/2\, ab)+(b-a)^2$ [19].

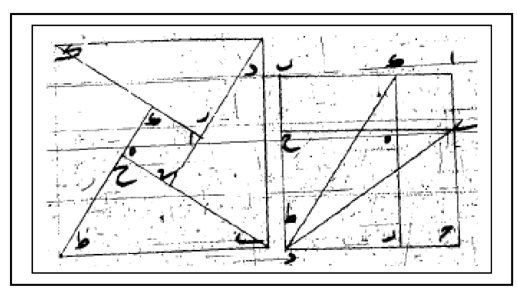
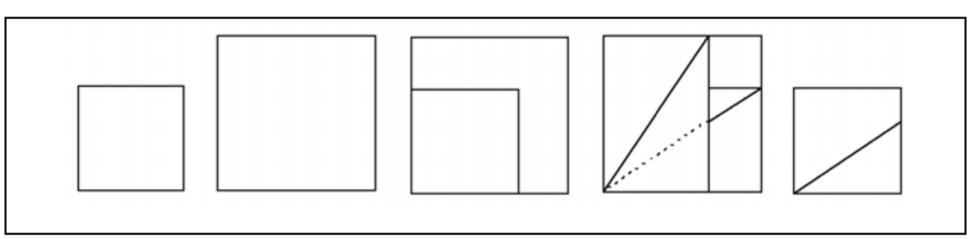


Figure 6: *Generating a square from two different size squares extracted from the original document.*



(*a*) (*b*) (*c*) (*d*) (*e*)

Figure 7: *The details of generating a square from two different squares.*

The above problem inspired me to think about creating a hinged game based on this dissection. After trying different sizes of squares, I realized that the case would come to a result if the special case for $a = \frac{1}{2}b$ is considered. Then, in fact, I can drop the small square and think about a hinged game that from one side shows a solid square of side $2b$, and from the other side presents a punched square with a square hole of side b in the middle, as in Figure 8.

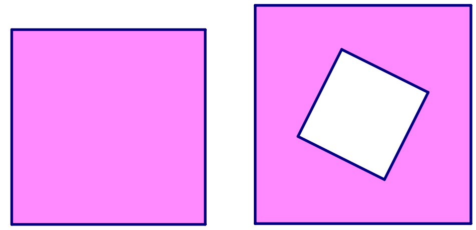


Figure 8: *The 4-square 5-square hinged game.*

This means that a square with area $4b^2$ can be changed to a square with area $5b^2$ that contains a hole with area b^2.

The following figure is a swing-hinged dissection of the above problem. I used circles to decorate the squares in a way that the dissection presents meaningful patterns when swung between the two squares.

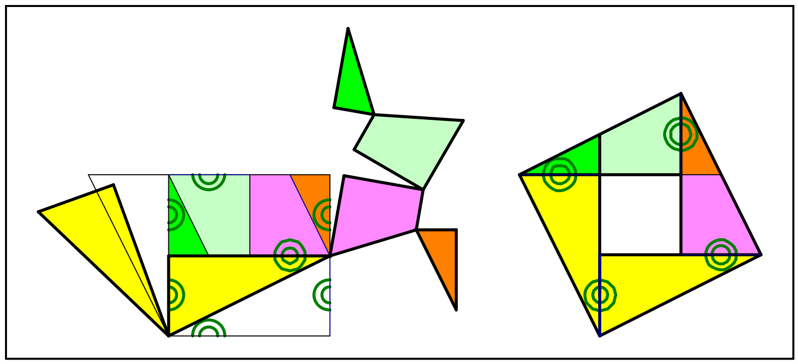


Figure 9: *The details of a solution, along with the circles.*

In order to see the designs in an appropriate position, each image was properly rotated. To honor Abûl-Wefâ for presenting the idea for this hinged game, the author named the following the *Abûl-Wefâ's Dual Design*.

To appreciate Figure 10, which is similar to the approach in the *Loyd's Sedan Chair Puzzle* presented in *Dissections, Plane & Fancy* [7], which hides the cut lines to make a puzzle, we may consider this figure as a puzzle, and ask for its hinged dissections. This would be a challenging puzzle.

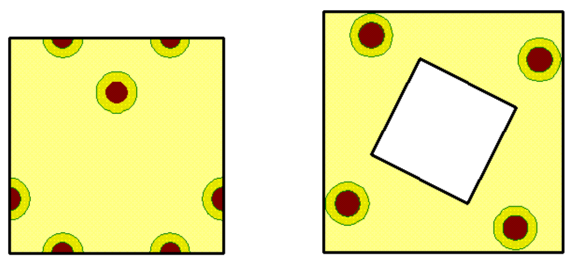


Figure 10: *The Abûl-Wefâ's Dual Design.*

The book shows another dissection, for the general case of an arbitrary small square hole, which was contributed by Henry Perigal [18]. Figure 11 presents both the Perigal's dissection and a Dual Design based on Perigal's dissection

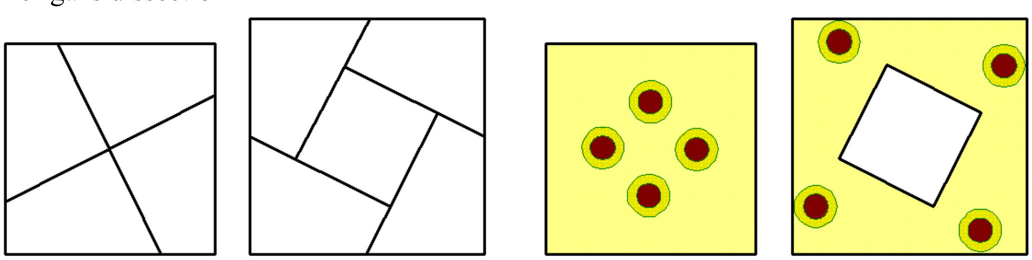
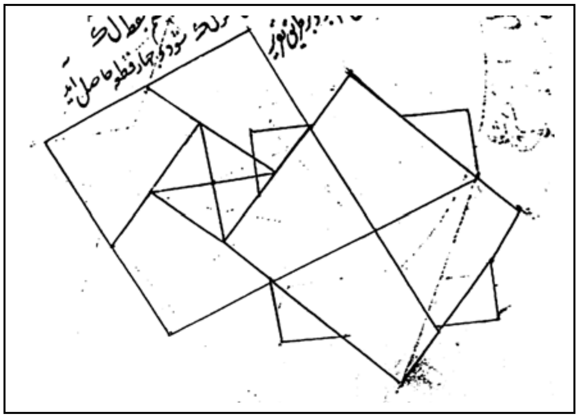


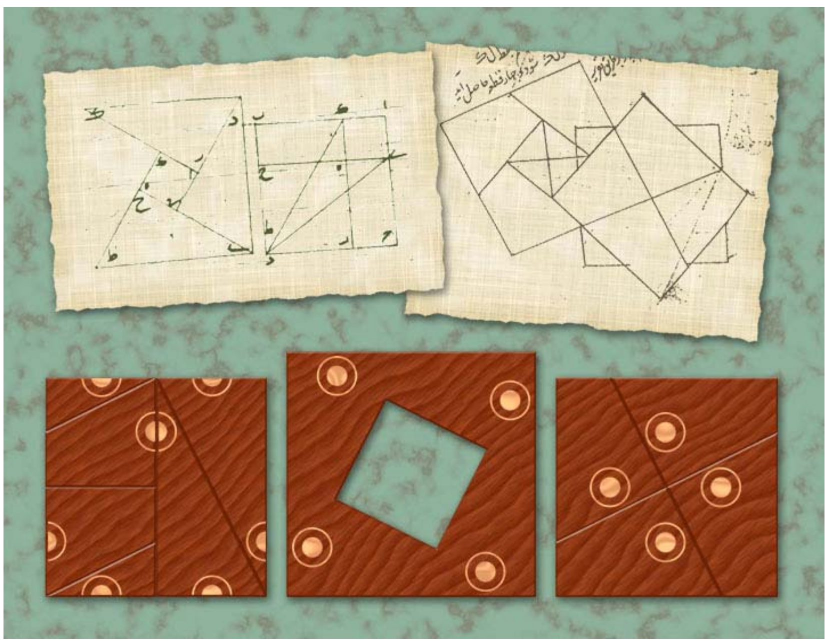
Figure 11: *Perigal's Dissection and its Dual Designs.*



Before leaving this section, there is a dissection design in the *Interlock* document [2] that strikingly resembles the Perigal dissection. Nevertheless, the purpose of this dissection is to transform a square to an eight-pointed star and has different components (Figure 12).

This star can be constructed by superposing a square over another congruent square, which has the same center, but has been rotated 45°.

Figure 12: *The image is extracted from the original document.*



The artwork in Figure 13 by Reza Sarhangi and Robert Fathauer presents the two mentioned puzzles on the lower half and the two original medieval dissections on the upper half.

Figure 13: *From Buzjani to Perigal*

3. Interesting Dual Designs on the Regular Triangle and Hexagram Discussion

Figure 14 demonstrates a 6-piece dissection of a regular triangle-hexagram that is illustrated in the *Interlock* document, a seven hundred years old Persian mathematics document [2]. Alpay mentions that the author of this document is anonymous [17]. However, in a recent Persian book, which includes the modern translations of both Abûl-Wefâ‘s treatise and this document, the author was identified as the Persian mathematician Abûl-Es-hâgh Koobnâni [12]. This dissection is not hingeable.

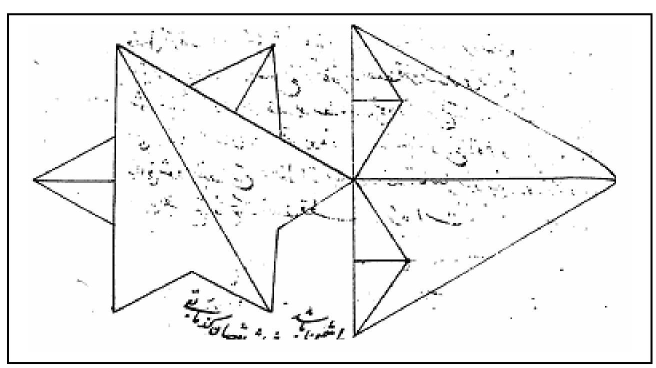


Figure 14

The 5-piece hinged dissection in Figure 15 is from Geoffrey Mott-Smith [16]. The *Hinged Dissections* book indicates that neither Mott-Smith nor Lindgren, who also created this dissection, identified it as hingeable. The book then illustrates one of the five possible cases of the hingeable triangle-to-hexagram dissection [8].

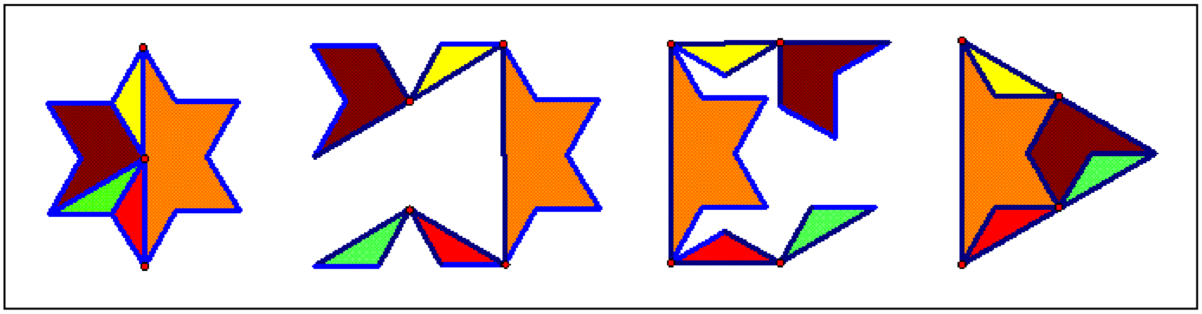


Figure 15

Using the above triangle-to-hexagram dissection, the author found the following changeable patterns (See Figure 16). The patterns may seem very natural and therefore obvious to the observer. However, to discover them was a different story. An important guide for the discovery of the patterns was the study of the fundamental region for each. With proper cuts, the hexagram can be divided into six rhombuses. Each rhombus can be divided into four right triangles. Each of these triangles, as the fundamental region, may generate the entire shape.

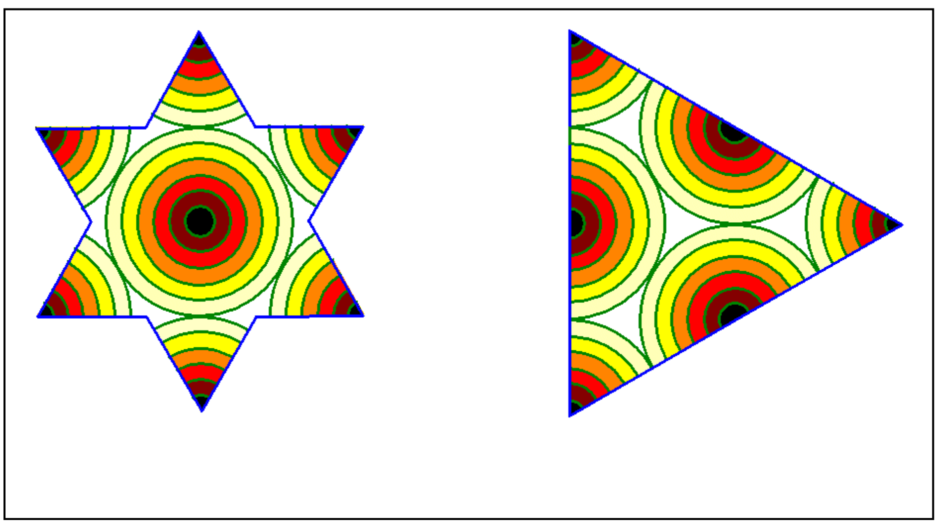


Figure 16

4. Tessellation: Technique for Making Dissections, Techniques for Making Designs

In Chapter 3 of the book *Hinged Dissection*, a powerful technique for dissection is presented. The technique is to superpose two tessellations in a way that the common pattern of repetition is preserved. The lines of one tessellation become the cuts for the other tessellation. The book continues to discuss which dissections are also hingeable. Figure 17 includes the Greek Cross-to-Square hingeable dissection tessellation presented in Chapter 10 of *Dissections, Plane & Fancy*. Moreover, it includes the patterns on the surfaces of the tiles that I created using this tessellation.

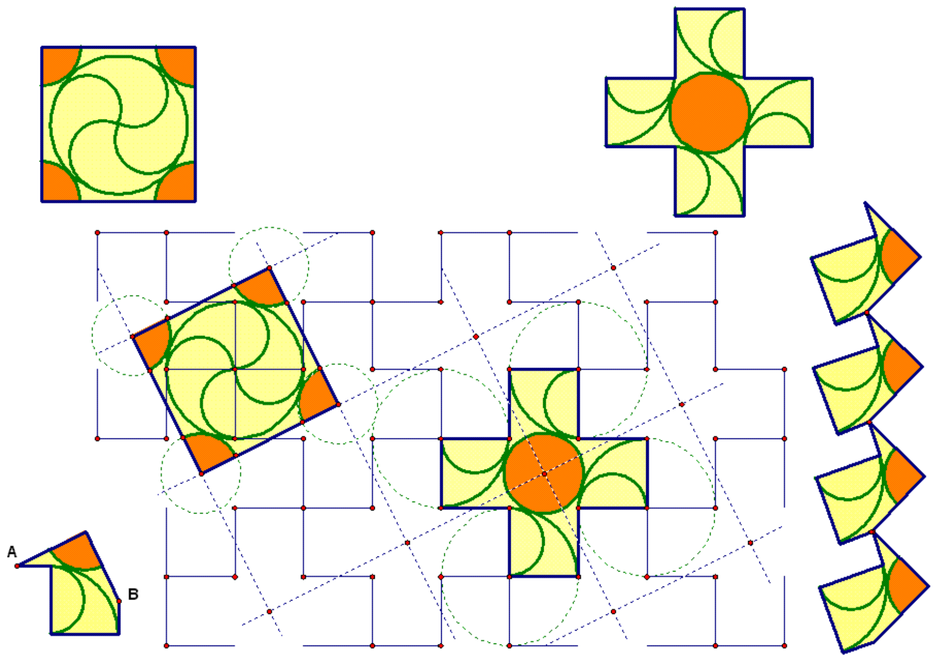


Figure 17: *This tessellation presents the necessary cuts for the Greek Cross-to-Square dissection. The tessellation also presents the way that the patterns have been created. On the left, a unit of the dissection is shown, with hinges at A and B. On the right, the unfolded hinged dissection is shown. At the top, the two pleasing patterns based on the Greek Cross-to-Square dissection are shown.*

5. Making Designs on the T-Strip Disection

The book *Hinged Dissection* introduces another technique for cases where the tessellation does not work, such as the triangle-to-square one:

> Harry Lindgren [13] showed how to derive it from the crossposition of two T-strips. In this method, we cut a figure into pieces that form a strip element. We then fit copies of this element together to form a strip but rotate every second element in the strip by 180° and match it with an unrotated twin. Thus we can call it a *twinned-strip*, or *T-strip*. Since every second element is rotated, every two consecutive elements in the strip share a point of 2-fold rotational symmetry. Let's call such points *anchor points*. We can similarly create a T-strip for the other figure. We then crosspose the two T-strips, making sure that an anchor point in one strip that is covered by the other strip either overlays an anchor point in the other strip or falls on a boundary edge of the other strip.

Figure 18 for the Kelland's gnomon-to-square dissection illustrates the method mentioned above.

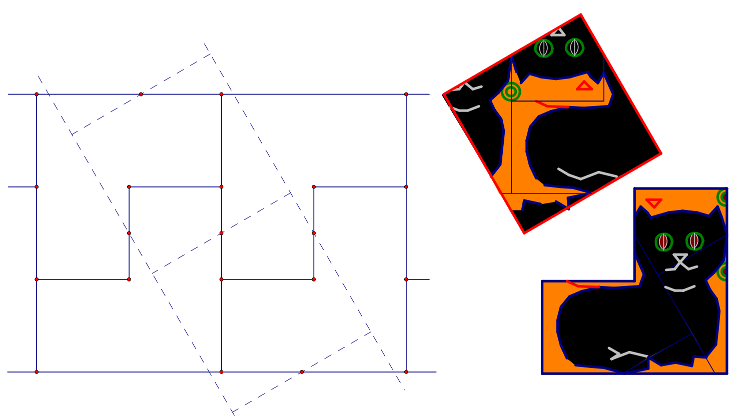


Figure 18: *The Transformation of a cat to a camel's head.*

Conclusion

Dissection combines art, mathematics, and entertainment by peeking the interest and fascination of cultures throughout mankind's history. This paper presents ideas to make the subject matter even more interesting and more imaginative. By bringing the dissections into the world of art, by decorating the pieces with patterns that change in a pleasing way, we are able to extend the horizon of the topic in a new and intriguing direction.

References

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