Some Monohedral Tilings Derived From Regular Polygons

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Abstract

Some tiles derived from regular polygons can produce spiral tilings of the plane [1]. This paper considers some more general classes of tilings with tiles derived from regular polygons, some have central symmetry, many have periodic symmetry, some have both, and a few have no symmetry at all. Any of these tiling patterns could be the basis for some interesting mathematical art, for example by colouring or decorating the tiles.

Using Regular Polygons to Make Tiles

Overlapping copies of a regular polygon around a common vertex produces a pattern of rhombi. A tile can be chosen bounded by sections of the perimeters of three of the poygons (figure 1). A section of the perimeter of a polygon will be termed a *polygonal arc.*

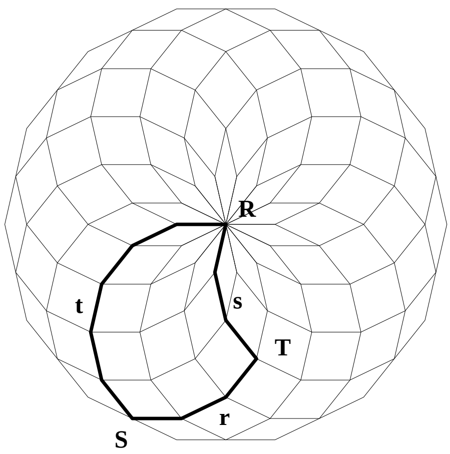


Figure 1. *The construction of a tile.*

If the polygon is a regular n-gon then the angles of the rhombi are $1/n$, $2/n$, $3/n$ ... (measured as fractions of a full rotation; multiply by 360 for degrees, or by 2π for radians). If the arcs are measured in units equal to a side of a polygon then by inspection:

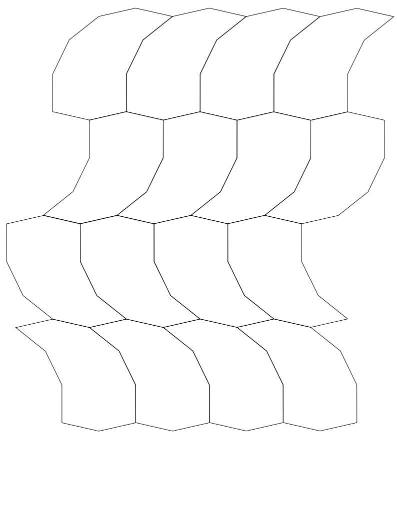
$$t = r + s$$

$$\angle R = r/n$$

$$\angle S = (t - 1)/n$$

$$\angle T = 1/2 - s/n$$

Figure 2. *Copies of any tile with r = 2 will tile the plane.*



The tiles can lie between the zig-zag lines in any of four orientations (reflected or rotated by 180•). The tiles meet at two types of vertex, where the angle sum = 1.

$$1/2 - 1/n + (t - 1)/n + 1/2 - s/n$$

$$= 1 + (t - s - 2)/n$$

$$= 1 + (r - 2)/n = 1$$

$$2(1/2 - 1/n) + r/n$$

$$= 1 - 2/n + r/n = 1$$

In general it will not be possible to tile the plane with tiles made like this because every convex section must be matched by a concave section, which can never occur. There are two special cases: if r = 1 then the arc is neither concave nor convex; if r = 2 then in all cases the tiles will fit as in figure 2.

If ∠ S is an exact fraction of a full rotation then a centrally symmetric tiling is possible (figure 3).

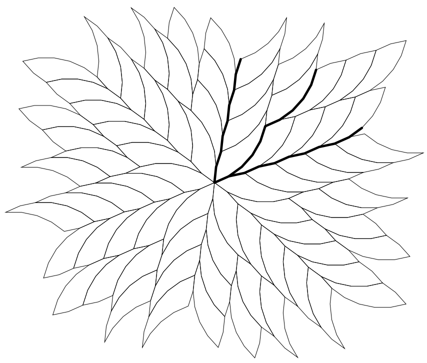
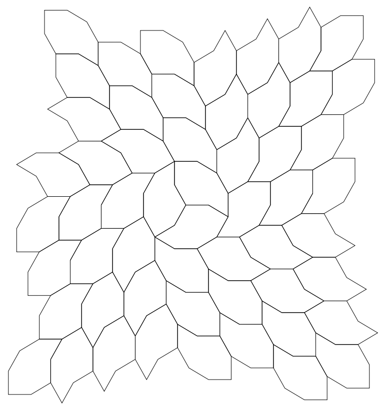


Figure 3. *A centrally symmetric tiling.*

The tiling consists of V-shaped sections with ∠ S at the apex. Each section is in two parts. The tiles in one part are mirror-images of the tiles in the other part. This is the underlying structure of all centrally-symmetric tilings with tiles derived from overlapping polygons.

There are some special cases of tilings based on this basic tile. One, found by Robert Reid, has n = 12 and s = 2, so t = 4, since r must be 2 (figure 4). Another has n = 10 and s = 3, so t = 5 and ∠ S = 2/5, the interior angle of a 10-gon, so the arcs r and t form a single arc of 7 sides. This tile has been investigated by Haresh Lalvani [2], and it allows a spiral tiling of the plane [3].

Figure 4. *A special case.*



In Robert Reid's tiling there is a central core that has rotational symmetry of order three, while the rest of the tiling has rotational symmetry order four, so the tiling has no overall symmetry.

As well as the usual V-shaped sections this tiling has four radiating "spokes".

Bi-concave Tiles

In figure 1, if the arc, r, is reflected in the line ST a new tile is produced. This is equivalent to removing a section from the tile that is bounded by two polygonal arcs that has apical angles of (r - 1)/n. The new tile has angles:

$$\angle R = r/n$$

$$\angle S = (t-1)/n - (r-1)/n$$

$$= (t-r)/n = s/n$$

$$\angle T = 1/2 - s/n - (r-1)/n$$

$$= 1/2 - (r+s-1)n = 1/2 - (t-1)/n$$

This time the length of the convex arc is equal to the combined length of the concave arcs, and all such tiles will tile the plane (figure 5).

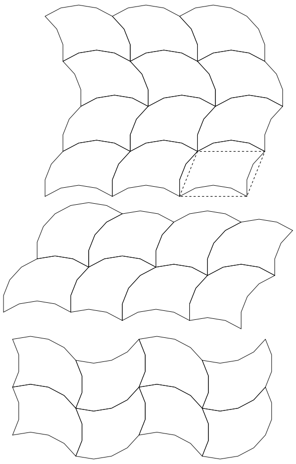


Figure 5. *All bi-concave tiles will tile the plane in various ways.*

We can see that any of these tiles will work because for any tile the angle sum at each vertex of the tiling is always 1.
The tile can be seen as a modified parallelogram.

$$\angle R + \angle S + \angle T + 1/2 - 1/n$$
$$= r/n + s/n + 1/2 - (t - 1)/n + 1/2 - 1/n$$
$$= 1 + (r + s - t)/n = 1$$

When $s = 2$ there are further possibilities because the tiles can fit together in a staggered fashion, analogous to figure 2.

If $\angle$ R is an exact fraction of a full rotation it is possible to produce a centrally symmetric tiling that consists of two-part V-shaped sections in a similar way to the example in figure 3. When both r and s divide n two centrally symmetric tilings are possible (figure 6).

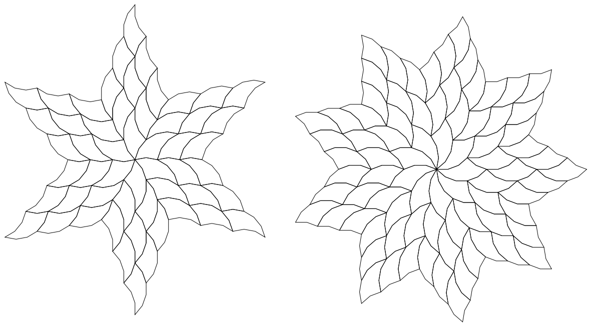


Figure 6. *Two centrally symmetric tilings using a tile with n = 18, r = 2, s = 3.*

The value of t is determined by r and s, and four tiles will always fit around a vertex if angles of each type are present (as in figure 5), so all such tilings with the same values of r/n and s/n have identical structures. In the limit the tiles will have sides that are circular arcs (provided that they have the same radius, and the arc lengths satisfy $t = r + s$). Provided that it does not depend on some specific number (such as the tilings like figure 1, when r must be 2) any tiling has an infinite series of equivalent tilings, including the limiting one with circular arcs. The general case of spiral tilings described in [1] is of this type, although the special cases (most of the examples) are not, since they depend on particular number properties.

An interesting example is provided by the three-armed spiral using a tile from a hexagon (figure 13 in [1]). Doubling the values of n, r and s gives a tile with $n = 12$, $r = 2$, $s = 4$, $t = 6$ with the special property that it can be dissected into two tiles with $n = 12$, $r = 1$, $s = 4$, $t = 5$. Two three-armed spirals that use this tile were described in [1], this is a third (figure 7).

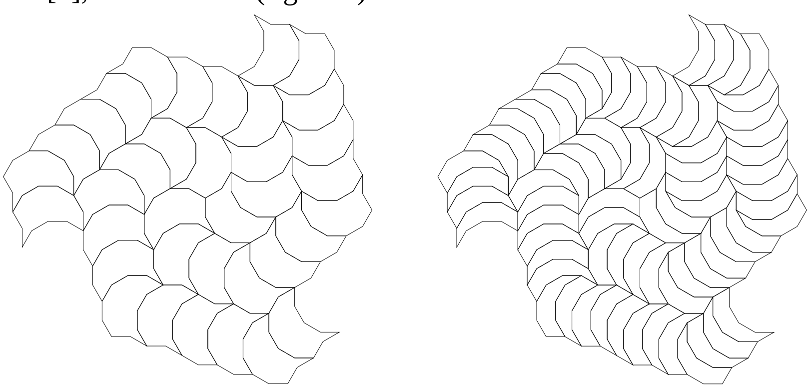


Figure 7. *A three-armed spiral tiling dissected.*

Special Cases

Figure 8 shows an additional way that some tiles can fit together. Usually there are no further tilings, apart from the cases when $s = 1$ and $s = 2$. If $s = 2$ the tiles can also fit in a way that is analogous to the tiling in figure 2 (see figure 9).

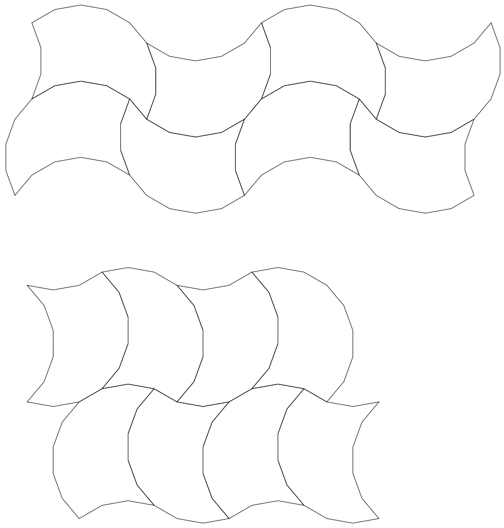


Figure 8. *An additional type of tiling.*

There are two types of vertex, both valency = 3.
The same condition is satisfied by both types

$$\angle R + \angle S = 1/2 - 1/n$$

$$r/n + s/n = t/n = 1/2 - 1/n$$

$$t = n/2 - 1$$

$$\angle T = 2/n$$

$$1/2 - (t - 1)/n = 2/n$$

$$t = n/2 - 1$$

The tile with $n = 10$, $r = 2$, $s = 2$ (so that $t = 4 = n/2 - 1$) is very unusual, and allows a wide range of tilings: standard tilings of the types already described (figure 9); figure 10 shows a few unusual ways that this tile can cover the plane; figure 11 shows a centrally symmetric tiling with V-shaped sections around a central core, and another that uses the same V-shaped units arranged asymmetrically; figure 12 shows a tiling that can be seen as spiral in two different ways.

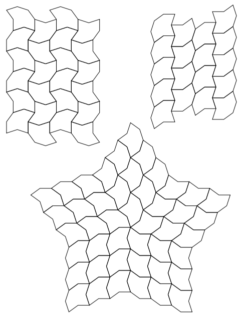


Figure 9. *Standard tilings using a special tile from a decagon.*

This tile has mirror symmetry, so there are fewer different tilings that follow the usual patterns. What are two different arrangements in figure 6 are identical, and the V-shaped sections in the centrally-symmetric tiling do not appear to be in two parts (as in figure 3). The variations in figure 5 all correspond to the single arrangement in a V-shaped section.

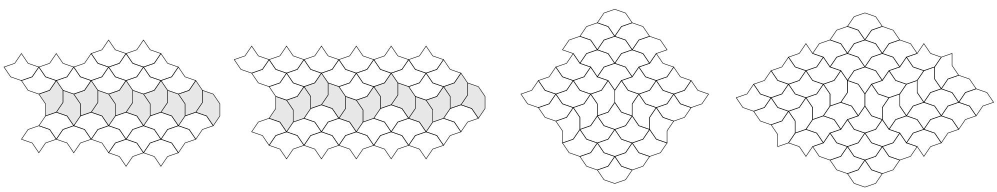


Figure 10. *Some unusual ways the special tile can cover the plane.*

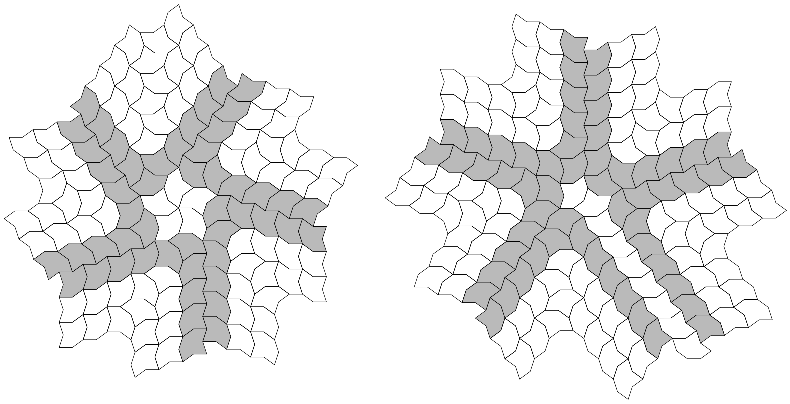


Figure 11. *A centrally symmetric tiling with five V-shaped sections, and another with the same sections.*

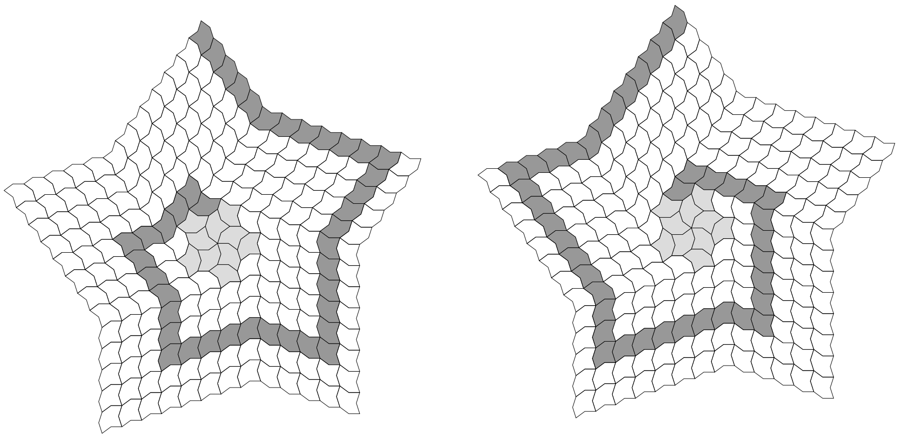


Figure 12. *A tiling that is spiral in two different ways.*

Periodic Tilings

There are two cases when central regions of centrally-symmetric tilings will themselves tile the plane periodically: when they are hexagonal or square, possibly modified. Figure 13 illustrates how the tile with n = 6, r = 1, s = 2, t = 3 can form hexagons of various sizes, but there are equivalent tilings with these values multiplied by any integer. As we have already seen (figure 7) the special case when they are doubled produces a tile that can be dissected into two smaller congruent tiles (n = 12, r = 1, s = 4, t = 5).

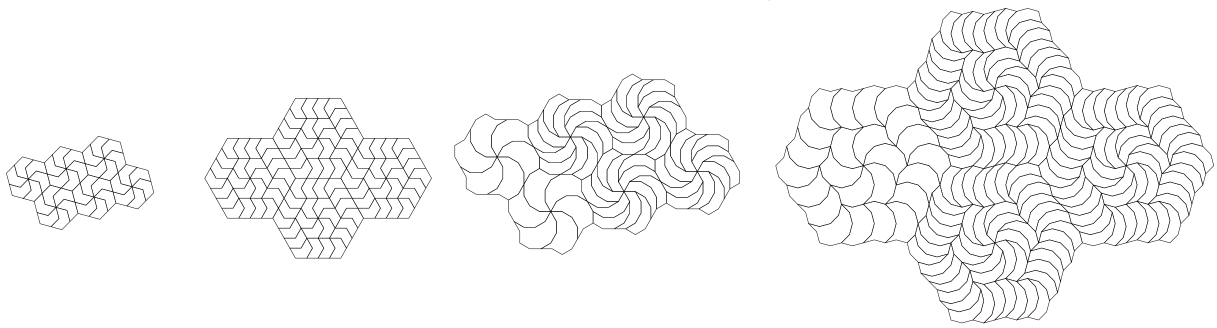


Figure 13. *Periodic tilings based on hexagons.*

The square itself is the basis for the other type of periodic tiling (n = 4, r = 1, s = 1, t = 2), and again the case when these values are doubled is special because it produces a tile that can be dissected into two smaller congruent tiles, but in two different ways, since it has mirror symmetry. This allows many variations of the basic tiling (figure 14).

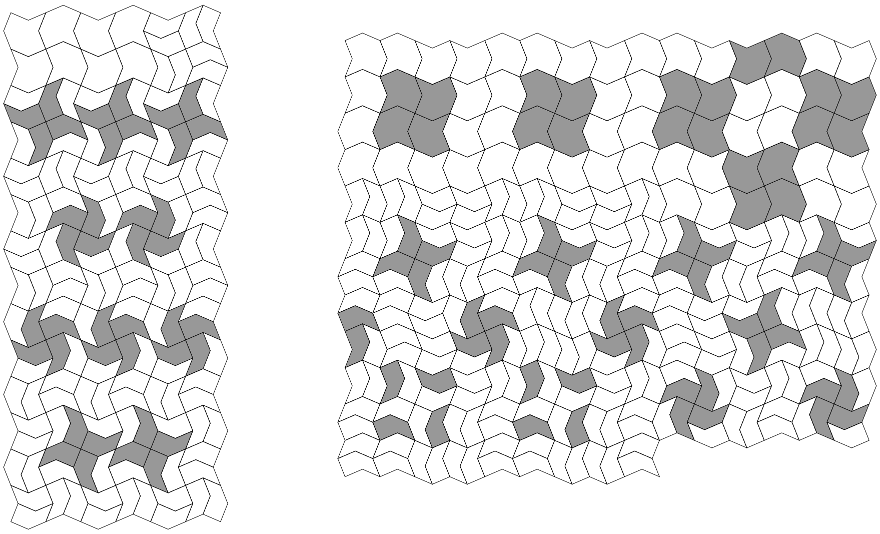


Figure 14. *Variations on periodic tilings based on squares.*

Further Possibilities

Although a few tilings with tiles having r = 1 have been considered there are probably many more to be found, since the constraint imposed by matching concave and convex arcs is relaxed. There has been no consideration of tilings derived from polygons having an odd number of sides, which seem to be more limited, but they are known to allow at least one type of spiral tiling [4], and many tilings using the reflex equilateral pentagon are known [5]. Versatiles [6] appear in a range of spiral tilings, but only the pentagonal versatile [7] seems to allow a particularly wide range of tilings, nevertheless there are probably more to be discovered. Another possibility yet to be explored in any detail is the use tiles with boundaries consisting of more than three polygonal arcs.

Many of these tilings are aesthetically pleasing as they are, and they can be coloured in many different ways. Figure 14 only begins to hint at some possibilities. The resulting patterns could have applications in architectural details (literally tilings), or in textile design, or simply as works of mathematical art in their own right.

References

[1] Gailiunas P., "Spiral Tilings" *Bridges 2000*, pp.133-140.
[2] US Patent 4,620,998, 1986, cited in Lalvani H., Meta Archtecture, in *Architecture and Science* (ed. Di Cristina G.), Wiley Academy, 2001.
[3] Stock D.L. and Wichmann B.A., "Odd Spiral Tilings", *Mathematics Magazine* vol.**73**, no.5, Dec 2000, pp.339-347.
[4] Hatch G. "Tessellations with Equilateral Reflex Polygons", *Mathematics Teaching*, no.84, Sept.1978, p.32.
[5] Simonds D.R., "Central Tessellations with an Equilateral Pentagon", *Mathematics Teaching*, no.81, Dec. 1977, p.36.
[6] Grünbaum B., and Shephard G.C., "Spiral Tilings and Versatiles", *Mathematics Teaching*, no.88, Sept. 1979, pp.50−1.
[7] Rice M. and Schattschneider D., "The Incredible Pentagonal Versatile", *Mathematics Teaching*, no.93, Dec. 1980, pp.52−3.