The Spirograph and Beyond

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Abstract

The advent of improved machine technology combined with interest in geometry and mathematics in the middle 19th century led to the creation of a variety of "mathematical machines" some of which were developed to create complex, intricate and beautiful designs. This paper takes a brief look at one of these machines as well as it's more contemporary namesake, and attempts to adapt some of the techniques using a modern drawing program to create additional designs.

20th Century Spirograph

Many mathematically-minded people will recall a drawing toy from their youth called the "Spirograph" that was produced by Kenner Co. It consisted of plastic wheels in a variety of sizes with uniformly notched edges and interior holes. There were also additional plastic "rings" notched on both the interior and exterior onto which the smaller wheels could be attached.

Placing the pen point into a hole and rotating the wheel about a stationary ring (it was pinned to the paper) would ideally produce a pleasing design of loops that varied according to the relative size of the rotating wheel and the stationary ring. Unfortunately sometimes the wheel would slip, often at the end, and spoil an otherwise lovely design. However, the potential existed for many exquisite, complex designs.

19th Century Spirograph

The London Science Museum has a wonderful display of "mathematical machines" from the past, among them several whose sole purpose was to create complex designs, often to be used on banknotes, stock certificates, or other documents to deter counterfeiting. Last summer on a tour of the LSM archives I was delighted to discover a machine called a Spirograph invented in Vienna in 1848 by a man named P. H. Desvignes. Accompanying it was a description and sample of curves drawn by the machine. See Figure 1.

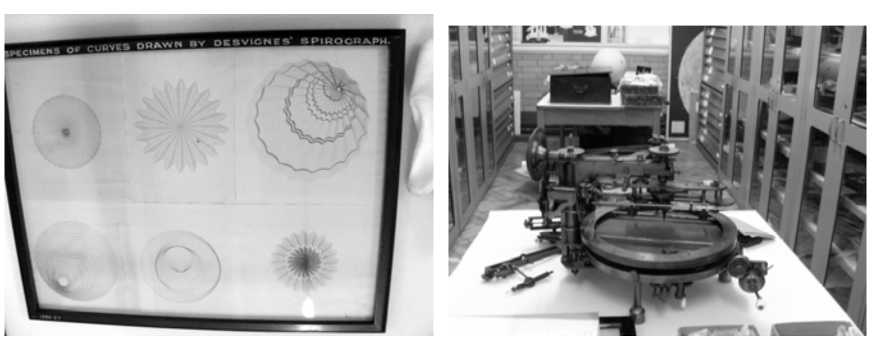


Figure 1: *Desvignes' 1848 Spirograph and designs*

This spirograph was made of carefully machined metal wheels and gears that drove a pen-mechanism and produced even more than, but including the designs made by the contemporary toy. Some of the examples shown above are spiral curves.

This raises the question, “using today’s technology can we duplicate the designs of the past, improve upon them, and can we develop brand new designs incorporating some of the techniques of the past?”

Roulettes

Of course mathematicians have known about curves for many centuries. They have studied them, analyzed them, catalogued and named them in an orderly fashion. The curves of the first Spirograph described and a subset of the second belong to the class of curves known as “roulettes”, or “the path of a point associated with a line or curve moving along another curve without slipping”. In particular consider a circle (C_w) rotating without slippage on the circumference of another fixed circle (C_f). Now think about the curve traced by a point on the circumference of C_w, by a point on the interior of (C_w), and by a point exterior to (C_w). All of these examples have names and are categorized.

If the rotating circle C_w is *exterior* to the fixed circle C_f the curves are called *epitrochoids*. If the point is *on* the circumference of C_w the curve is known as an epicycloid. If the point is on the interior of C_w the curve is a *cutate epiroulette*, and if the point is exterior to C_w it is called a *prolate epiroulette*. See Figure 2. [1]

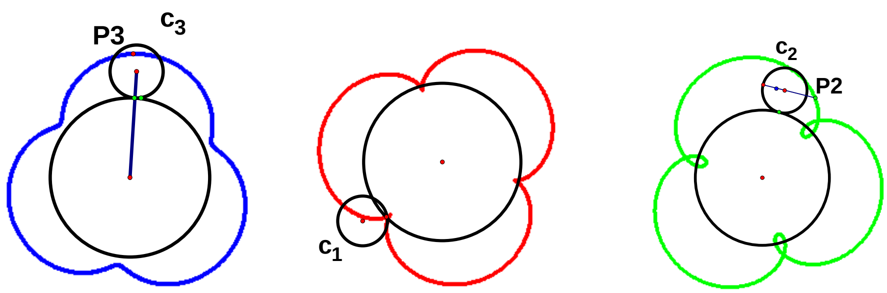


Figure 2: *Epitrochoids*

a) point on interior of C_w (cutate epiroulette) *b) on circumference (epicycloid)* *c) on exterior of C_w (prolate epiroulette)*

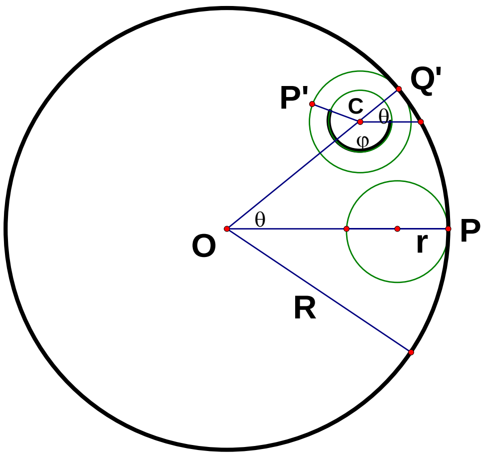
If the rotating circle C_w is *interior* to the fixed circle C_f the curves are called *hypotrochoids*. The corresponding terms are *hypocycloid* (point *on* C_w), *cutate hyporoulette* (point on the *interior* of C_w), and *prolate hyporoulette* (point *exterior* to C_w).

The Spirograph of the 20th century drew both epitrochoids and hypotrochoids, but could not draw the extended prolate versions.

Parametric Equations

How can we duplicate these curves? Equations for the epicycloid and hypocycloid are not too difficult to develop.

For a wheel with point P on the circumference rolling inside a circle (a hypocycloid) imagine the following representation. A circle of radius r with point P on the X-axis rolls clockwise within a circle of radius R to a new position P’ as in Figure 2. The original coordinates of P are (Q,0). [2]



Coordinates of P' relative to C are (r cos φ, -r sin φ)
Coordinates of C relative to O are ((R-r) cos θ, (R-r) sin θ)
Coordinates of P' relative to O are
x=(R-r) cos θ + r cos φ   y=(R-r) sin θ - r sin φ
But the distance Q'P = Q'P' or Rθ = r(θ + φ) or φ = [(R-r)/r]θ
Substituting this into eq. for x and y above,
x=(R-r) cos θ + r cos {[(R-r)/r]θ}
y=(R-r) sin θ - r sin {[(R-r)/r]θ}
These are the parametric equations for the <u>hypocycloid.</u>

Figure 3: *Derivation of parametric equations for the hypocycloid*

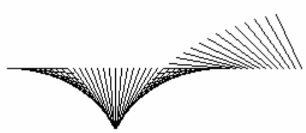
An analogous derivation for the *epicycloid* yields the equations

$$x = (R + r) \cos \theta + r \cos [(R + r)/r]\, \theta$$
$$y = (R + r) \sin \theta + r \sin [(R + r)/r]\, \theta.$$

Both of these sets of equations can be extended to cover a point P not on the rim of the turning circle. If h = distance from P´ to C then the second coefficient (r) should be replaced by h and the curves represented will be the hypotrochoids and the epitrochoids.

New Designs

Consider first one of the simplest cases, that of a line moving on another line in discrete steps, but rotating a small amount (5 degrees) at each position.



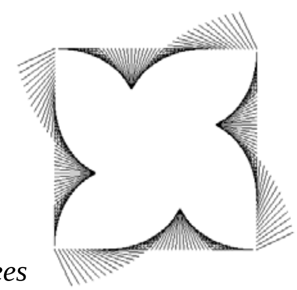


Figure 4: *Rotation = 5 degrees*

Varying the amount of rotation and repeating four times gives the following figures.

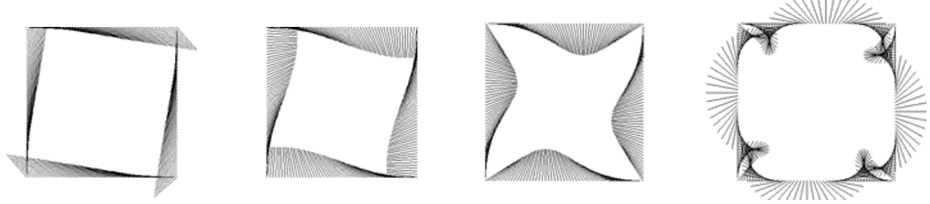


Figure 5: *Amount of rotation = 1, 2, 3, 10 degrees*

Using the Figure 4(b) as a standard, what does varying the length of each step along the base line do?

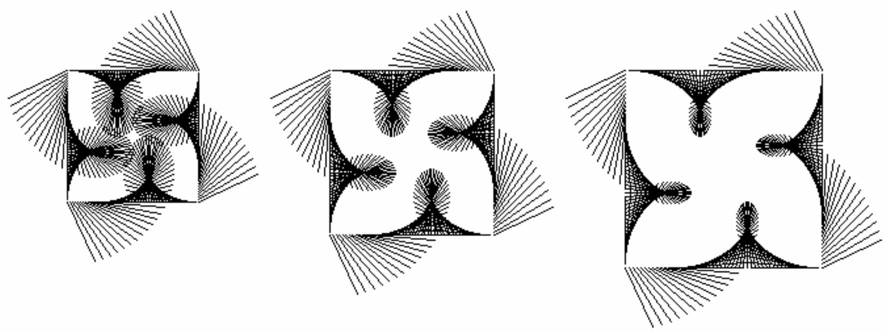


Figure 6: *Distance between steps = 2, 2.5, 3*

This obviously stretches the figure in two directions as well as decreases the size of the inner loops.

Choosing the number of repetitions times the rotation to be 180 leads to a symmetric figure which can then be repeated.

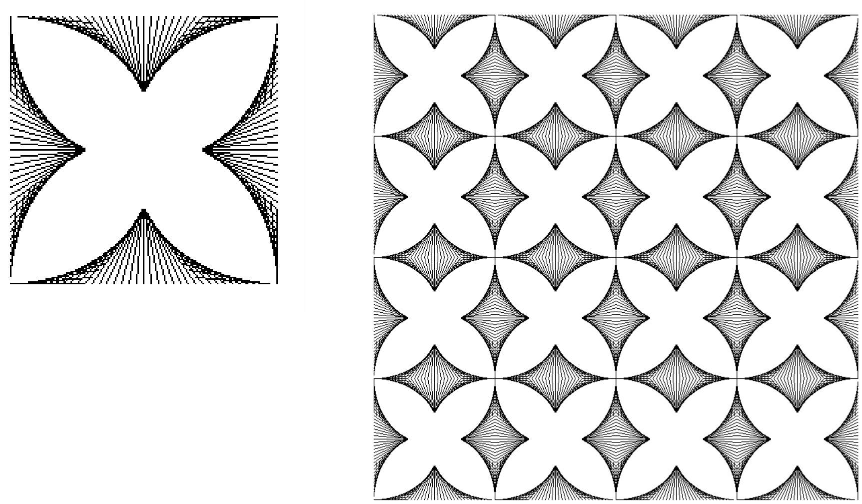


Figure 7: *Modular number of repeats*

Figure 7 is made of *only straight lines*, which form the curves when rotated. Finally, this pattern can be repeated and overlaid at different offsets to create a unique and complex design once again made of straight lines only (Figure 8). The "mountains" formed are reminiscent of the "ladder against the wall" problem where a ladder leaning against a wall traces a curve called an astroid as it falls to the ground. In this case the line approaches the wall so the curve traced is slightly different.

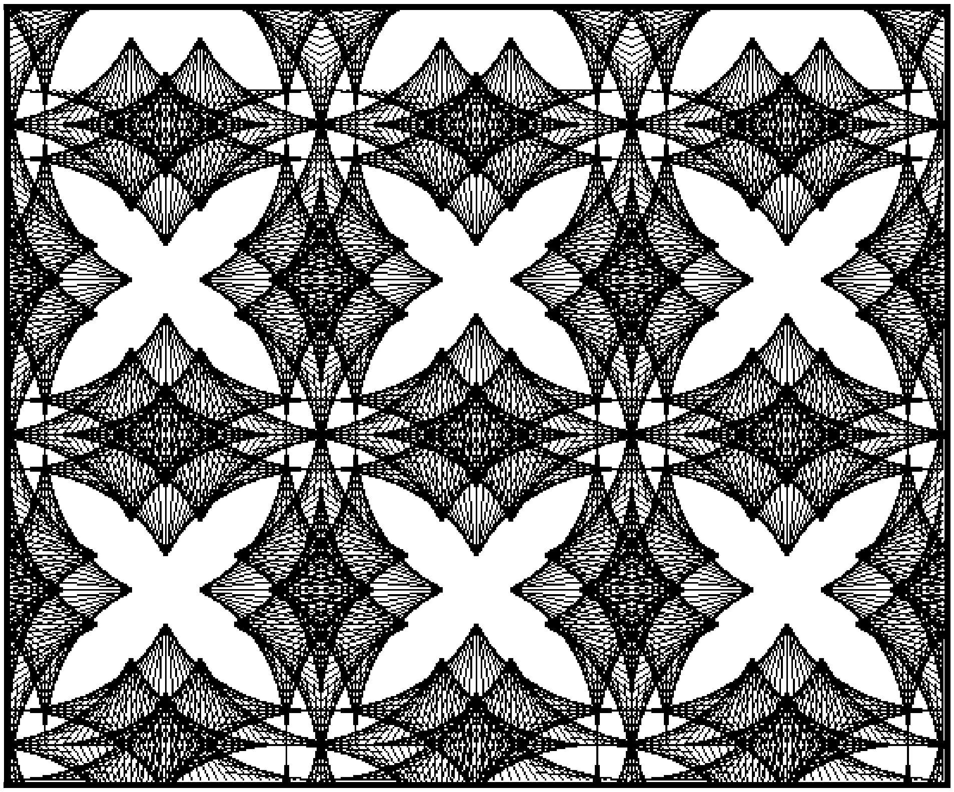


Figure 8: *Overlays of pattern in 7(b)*

Drawing repeated cycles of a particular shape was another common technique. Here that has been adapted using triangles and squares with rounded corners. Most surprising are the patterns formed by the interaction of the parts. No one could predict the lacy interior of the triangle design in Figure 9 or the circle that appears out of nowhere to encase the "petals".

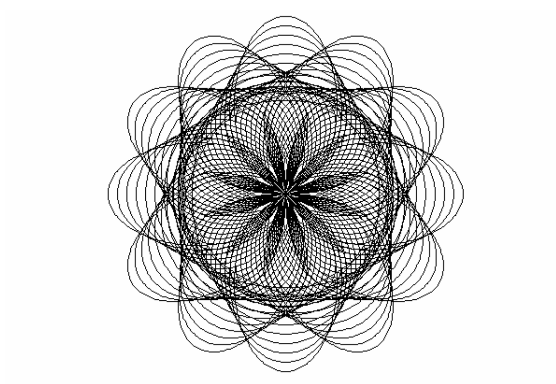


Figure 9: *Rounded triangles*

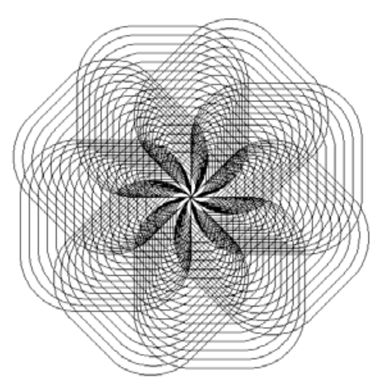


Figure 10: *Rounded squares*

Similarly, the rounded squares in Figure 10 form an unexpectedly delicate flower pattern with asymmetric petals which lend almost a pinwheel effect. The petal shapes are actually repeated in four concentric rings about the center with the innermost set being the negative space created by the squares. Each petal version is shaped a little differently, but with its neighbors forms its own flower shape.

Summary

These examples illustrate just some of the techniques from the past that can be adapted to today's computer environment. Stepwise drawing, repeated elements, layer overlays all had their place as well as continuous pen drawing influenced by external movement. With modern computer drawing programs it becomes possible to experiment and even move beyond the limitations of mechanical systems. There is obviously a rich area here to be explored with all the excitement of discovery that surely was experienced by the original makers of the mathematical drawing machines.

References

[1] Glassner, Andrew, *Morphs, Mallards, and Montages*, A K Peters, 2004
[2] Maor, Eli, *Trigonometric Delights*, Princeton University Press, 1998